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A new approach for fitting growth models in random environment

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ABSTRACT

Nonlinear growth models are widely employed in Animal sciences for describing growth of various species of animals. Nonlinear estimation procedures are generally employed for estimation of parameters. However, one limitation of these models is that they are applicable only when the data are available at equidistant epochs. Another limitation is that the fluctuations in the system cannot be satisfactorily explained simply by adding an error term to the deterministic formulation. The purpose of this article is to bring to the notice of Animal scientists the new approach of Stochastic differential equation modelling, which is capable of incorporating both the above aspects. The methodology is discussed by considering Gompertz growth model. Relevant SAS codes for fitting the model are developed. Finally, the methodology is illustrated on secondary monthly pig weight data, collected at the piggery farm of Indian Veterinary Research Institute, Izatnagar, Bareilly, India.

Key words: Gompertz growth model, Pig weight data, SAS software package, Stochastic differential equation model

Nonlinear growth models, such as Gompertz, Logistic, and von Bertalanffy models are widely employed in Animal sciences for describing growth of various species of animals. A heartening aspect of these models is that they are mechanistic in nature and so the underlying parameters have specific biological interpretations. For example, Intrinsic growth rate, Carrying capacity, and Initial population size are three parameters in Gompertz model. Growth models are generally expressed in terms of nonlinear differential equations. An attractive feature of these models is that they can be converted to linear forms by means of some transformations, like Logarithmic and Reciprocal. Consequently, exact solutions of the underlying differential equations can be obtained, which are nonlinear in parameters. Usual practice for applying them to data is to add an additive term, with suitable assumptions, on the right hand side of the deterministic solution and applying Nonlinear estimation procedures, such as Levenberg-Marquardt procedure (Seber and Wild 2003) for estimation of parameters. A large number of articles dealing with this methodology have appeared in research journals during the last two decades or so (Venugopalan and Prajneshu 1997, Prajneshu and Kandala 2003, Prajneshu and Ravichandran 2003, Matis *et al.* 2009, and Ghosh *et al.* 2011).

Although the above methodology has served many purposes in the past, it suffers some limitations. The first one is that it is applicable only when the data are equidistant. The philosophy behind above growth models is that the growth rate is fast in the initial phase and then it slows down in the next phase, thereby leading to a sigmoid type of curve. Therefore, quite often, animal scientists record growth data in the initial phase at quick intervals and in the subsequent phase at wide intervals. This leads to generation of data at unequal intervals. The second limitation is that, fitting a nonlinear deterministic model by simply adding an error term is not capable of describing the underlying fluctuations of the system satisfactorily, particularly for longitudinal data (Seber and Wild 2003). It may be highlighted that both the above limitations can be successfully tackled by employing the more general approach of 'Stochastic Differential Equation (SDE)' (Oksendal 2003). These are generally obtained by adding a stochastic term on the right hand side of the differential equation form of deterministic formulation of a growth model. It may be noted that, in a physical situation, random environmental fluctuations due to variations in parameters, such as birth and death rates generally occur with great rapidity as compared to the time-scale of population growth. Therefore, the stochastic term is generally assumed to be a Gaussian white noise stochastic process. A heartening aspect of this prescription is that the resultant process becomes Markovian. However, the price to be paid is that the sample paths are very irregular and do not admit of derivatives in the conventional sense. To handle this situation, two types of stochastic calculi due respectively to, Stratonovich and Itô, have been developed in the literature. In the former, usual rules of calculus continue to apply whereas in the

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latter, these are suitably modified. However, for the present article, both these calculi yield identical results as we shall deal only with models with additive noise, which is independent of state variable (See, e.g. Cohen and Elliott 2015). It may be noted that, in this article, we shall confine our attention to only studying the SDE version of Gompertz model.

MATERIALS AND METHODS

The differential equation form of Gompertz growth model (Seber and Wild 2003) is given by

$$dy_t/dt = ry_t\, log_e (K/y_t), \qquad (1)$$

Using logarithmic transformation, i.e. $g(y_t) = logy_t$, above deterministic nonlinear differential equation may be linearized. By inversion, its solution is

$$y_t = Kexp\, [log_e (y_0/K)\, exp\, (-r t)], \qquad (2)$$

The corresponding Gompertz nonlinear statistical growth model is

$$y_t = Kexp\, [log_e (y_0/K)\, exp\, (-rt)] + \varepsilon_t, \qquad (3)$$

where ε_t are independent and identically distributed (i..i.d.) with zero mean and variance σ^2. As it is logical that error in y_t would be proportional to the expected magnitude $\mu_t = Kexp\, [log_e (y_0/K)\, exp\, (-rt)]$, it follows that logarithmic transformation is the appropriate transformation for variance stabilization, which is required for efficient estimation of parameters. Accordingly, the transformed Gompertz statistical growth model is

$$log_e y_t = [log_e K + log_e (y_0/K)\, exp\, (-rt)] + \varepsilon_t^*, \qquad (4)$$

Following along similar lines as Filipe *et al.* (2013), analogous Gompertz SDE model with constant diffusion coefficient is given by

$$dZ_t = r(\alpha - Z_t)dt + \sigma dW_t, \qquad (5)$$

where $\alpha = log_e K$, $Z_t = log_e y_t$, and W_t is a Wiener process with variance parameter unity. Eq. (4) is equivalent to

$$d\exp(rt)Z_t = r\alpha dt + \sigma dW_t,\ Z_{t_0} = z_0.$$

After integration of both sides and applying Ito calculus, solution Z_t of the SDE model given $F_{t_k} = \{Z_{t_j} : j \le k\}$ is

$$Z_t = \alpha + (Z_{t_k} - \alpha)\, e^{-r(t-t_k)} + \sigma exp(-rt) \int_{t_k}^{t} exp(rs)\ dW_s. \qquad (6)$$

Note that solution of the above Gompertz SDE model given by eq. (6) is a Gaussian process with conditional mean and variance given respectively by

$$\mu_{Z:t|t_k} = E\{Z_t|F_{t_k}\} = \alpha + (Z_{t_k} - \alpha)\, e^{-r(t-t_k)},\quad \sigma^2_{Z:t|t_k} = V\{y_t|F_{t_k}\} = \frac{\sigma^2(1-e^{-2r(t-t_k)})}{2r}. \qquad (7)$$

The transition probability density function of the process $\{Z_t,\ t \ge 0\}$ is asymptotically stationary with mean and variance given respectively by α and $\sigma^2/(2r)$. However, it may be noted that the mean-value function of $\{Z_t\}$, i.e. $E[Z_t] = \alpha + (Z_{t_0} - \alpha)\, e^{-r(t-t_k)}$ is a sigmoid curve. It may be pointed out that the solution of Gompertz SDE model given by eq. (6) is capable to model growth under dependent error processes $\{\sigma exp(-rt) \int_{t_0}^{t} exp(rs)\ dW s, t \ge 0\}$. Finally, using moment-generating function of Gaussian random variable, predicted value of y_t may be obtained by evaluating conditional mean of y_t given past values of the process $\{y t_j : j \le k\}$, which is given by

$$\mu_{y:t|t_k} = exp\, (\mu_{Z:t|t_k} + 0.5\, \sigma^2_{Z:t|t_k}) \qquad (8)$$

The parameters of Gompertz SDE model in eq. (5) may be estimated by the Method of maximum likelihood, which is carried out by maximizing joint likelihood of the transformed process $\{Z_t\}$. To this end, joint likelihood is expressed in terms of product of conditional likelihoods at time-epoch t given F_{t_k}, which are Gaussian with conditional means and variances given by eq. (7). Relevant computer code in SAS software package for fitting the model to data is developed and the same is appended as Annexure-1. Finally, goodness-of-fit of animal-wise fitted model is assessed by computing RMSE, given by

$$RMSE = [\textstyle\sum_{t=1}^{n} (y_t - \hat{y}_t)^2 / n]^{1/2}, \qquad (9)$$

where y_t and $\hat{y}_t$ denote respectively the observed and fitted values at time t, and n indicates the number of time-epochs. Averages of above animal-wise RMSE values would reflect the overall performance of fitted model.

RESULTS AND DISCUSSION

As an illustration of the above methodology, monthly pig weight data, reported in Das (2015) and collected at the piggery farm of Indian Veterinary Research Institute, Izatnagar, Bareilly, India, are considered. The weights (kgs)

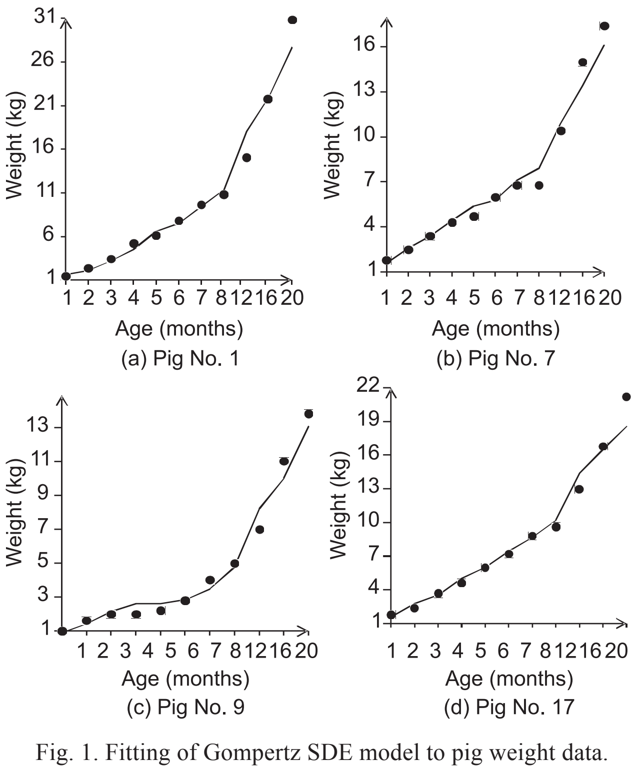

Fig. 1. Fitting of Gompertz SDE model to pig weight data.

Annexure-1

SAS Code for Gompertz SDE Modelling

```
%macro data;
ods trace on;
proc optmodel;
ods output PrintTable=parms_&kk.;
set l={n1..n2};/* User Defined Values*/
set j={n1+1..n3};
set k={n3+1..n4};
number y{l};
read data abc_&kk. into [_n_] y;
number n init n2;
var z{1..3}>=0;/* Optimization Variables*/
max f=sum{i in j}log((sqrt((z[1]/(2*z[2]))*(1-exp(-2*z[2])))**(-1)))-sum{i in j}(((z[1]/z[2])*(1-exp(-2*z[2])))**(-1))*((y[i]-(z[3]+(y[i-
1]-z[3])*exp(-z[2])))**2)+(sum{i in k}log((sqrt((z[1]/(2*z[2]))*(1-exp(-8*z[2])))**(-1)))
-sum{i in k}(((z[1]/z[2])*(1-exp(-8*z[2])))**(-1))*((y[i]-(z[3]+(y[i-1]-z[3])*exp(-4*z[2])))**2));
solve;
print z;
run;
ods trace off;quit;
%mend;
%macro iml;
proc iml;
x={DATA FILE};
%do i=1%to SizeOfData;/*if you want increase the data, then accordingly variable changes */
y1=x[,&i];
y_&i.=log(y1);
print y_&i.;
varnames={y};
create abc_&i. from y_&i.[colname=varnames];
append from y_&i.;
close abc_&i.;
%end;
create x1 from x;
append from x;
close x1;
%do i=1%to SizeOfData;/*if you want increase the data, then accordingly variable changes*/
%pigdata;%end;
proc iml;
%do i=1%to SizeOfData;/*if you want increase the data, then accordingly variable changes*/
use abc_&kk.;
read all into y_&kk.;
use parms_&kk.;
read all into z_&kk.;
use x1;
read all into x;
zz=z_&kk.[,2];
zz1=zz1||zz;%end;
print zz1;
%mend;%iml;
```

Table 1. Weight (kg) of pigs at various ages (months)

Age/Pig No	1	2	3	4	5	6	7	8	12	16	20	24
1	1.4	2.3	3.4	5.2	6.0	7.8	9.6	10.8	15.0	21.8	30.8	35.0
2	1.3	1.7	2.4	3.0	3.0	3.5	5.1	6.5	10.5	17.0	32.0	37.0
3	1.3	2.1	3.5	4.4	5.4	6.0	6.6	7.2	10.8	13.0	14.0	15.0
4	1.8	2.7	4.0	5.9	6.3	6.8	7.4	8.6	11.8	14.0	13.4	13.4
5	1.7	2.6	4.9	6.3	7.0	6.8	7.4	8.0	11.0	14.0	16.6	18.4
6	1.4	2.2	4.5	5.6	6.8	7.4	8.0	8.6	10.0	13.0	17.0	16.4
7	1.8	2.5	3.4	4.3	4.7	6.0	6.8	6.8	10.4	15.0	17.4	19.4
8	2.1	3.0	4.1	5.2	5.6	7.2	8.2	9.2	10.8	13.8	19.4	20.0
9	1.0	1.6	2.0	2.0	2.2	2.8	4.0	5.0	7.0	11.0	13.8	18.0
10	2.0	3.0	3.6	4.0	4.8	4.4	5.0	6.4	7.4	10.4	16.4	15.6
11	1.2	2.2	3.0	3.2	4.0	4.2	4.7	5.2	6.4	9.8	15.0	13.6
12	1.7	2.7	3.8	4.7	5.5	7.0	7.8	8.6	13.8	18.6	23.0	23.0
13	2.2	2.6	2.7	3.5	4.6	6.0	7.5	9.8	13.6	20.0	24.0	28.0
14	1.2	2.2	3.0	3.2	4.0	4.2	4.7	5.2	6.4	9.8	15.0	13.6
15	2.1	2.6	3.3	5.4	6.3	7.6	8.6	8.2	11.8	19.2	30.0	35.0
16	1.5	2.2	2.8	3.0	4.3	3.8	4.1	4.8	7.0	15.0	20.4	24.4
17	1.8	2.4	3.7	4.6	6.0	7.2	8.8	9.6	13.0	16.8	21.2	21.0
18	1.6	2.3	3.1	4.3	5.1	6.0	6.6	7.8	12.8	18.6	29.0	33.0
19	2.0	2.8	4.3	5.2	6.1	7.4	8.2	9.4	13.2	16.4	18.4	21.0
20	1.2	2.1	3.0	4.0	4.8	5.6	6.6	7.8	10.8	19.4	27.0	32.0

Table 2. Estimation of parameters for fitted Gompertz SDE model

Parameters/Pig No.	α	r	σ^2
1	3.67	0.12	0.008
2	4.03	0.09	0.019
3	2.60	0.23	0.004
4	2.55	0.29	0.008
5	2.63	0.28	0.017
6	2.64	0.27	0.017
7	2.83	0.18	0.008
8	2.72	0.26	0.016
9	2.77	0.13	0.016
10	2.45	0.24	0.032
11	2.34	0.26	0.028
12	3.04	0.20	0.008
13	3.30	0.14	0.020
14	2.34	0.26	0.028
15	3.58	0.12	0.016
16	3.25	0.11	0.024
17	2.99	0.20	0.007
18	3.38	0.15	0.010
19	2.81	0.24	0.009
20	3.42	0.13	0.008

of each of 210 pigs are observed at ages 0,1,2,…,8 months and thereafter at ages of 12,16,20, and 24 months. A random sample of 20 pigs is selected and the data are reported in Table 1. The data up to the age 20 months are used for fitting of Gompertz SDE growth model, while the data for age 24 months are employed for studying performance of the fitted model for prediction purpose. The performance of the fitted model is studied by computing mean square error values for each of 20 selected pigs. Using the developed SAS code in Annexure-1, estimates of the parameters of Gompertz SDE model are obtained and the same are reported in Table 2. Further, using eq. (9), average of RMSE values is computed as 1.35 Kgs., which being quite low, shows that Gompertz SDE model provides a good fit to the given data. To get a visual idea, the graphs of fitted models along with data are exhibited in Fig. 1 for 4 randomly selected pigs. Subsequently, performance of fitted model is assessed for prediction purpose by first computing pig-wise one-step ahead predictions and then computing the averages of pig-wise squared prediction error values by using eq. (9). This value computed as 2.25 Kgs., again reasonably quite low, reflects that performance of fitted model is reasonably good even for prediction purpose.

To conclude, it is hoped that animal scientists would now start applying the new approach for Stochastic differential equation modelling to their data sets.

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