

General Method of Construction of Constant Block Sum PBIB Designs With Two Replicates

Ravindra Khattree

*Department of Mathematics and Statistics, Institute for Data Science,
Oakland University, Rochester, MI 48309, USA*

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SUMMARY

A recent article by Yadav *et al.* (2024) presents two series of the constant block sum partially balanced incomplete block designs for number of treatments $\nu = 4n + 2$ and $\nu = 4n + 4$. The important feature was that the number of replicates $r = 2$ and pairwise replication parameters $(\lambda_1, \lambda_2, \lambda_3) = (2, 1, 0)$. In the present article, it has been shown that with the same parameters values of r and λ_i s, such restrictions can be relaxed and with the help of (nearly) magic rectangles, a much larger choice of designs can be made available.

Keywords: Constant block-sum designs; Magic rectangles; Nearly magic rectangles; Partially balanced incomplete block designs.

1. INTRODUCTION

In a recent article, Yadav *et al.* (2024) have given two series of constant block-sum designs for the number of treatments $\nu = 4n + 2$ and $\nu = 4n + 4$. The common feature in these designs is that the number of replicates is $r = 2$ and the number of blocks is $b = \nu / 2$. In fact, one can generate many more such designs and one does not have to confine to the above choice of ν being even. Consequently, b also does not have to be half of ν . The trick lies in using the magic squares, magic rectangles or nearly magic rectangles in the construction.

2. THE CONSTRUCTION

It is well known (for example, see Khattree (2023)) that a $p \times q$ magic rectangle always exists as long as (p, q) are in parity, that is, both are either odd or both are even. In case, one of these is odd and other even, a nearly magic rectangle can always be found. Numerous approaches are widely available to generate magic

squares and rectangles. An extensive list of nearly magic rectangles is also given in Khattree (2023) and a SAS code to generate these and many more is available as a supplement of the same article.

Without much ado, I will cryptically state the method to generate these designs. The essential common feature, as in Yadav *et al.* (2024), is that $r = 2$ and $(\lambda_1, \lambda_2, \lambda_3) = (2, 1, 0)$. The algorithm is simple enough to state in just four steps.

1. Take a $p \times q$ magic rectangle $\mathcal{M}_1$ (or a nearly magic rectangle where if p is even then consecutive rows must be appropriately arranged so that these consecutive rows have different sums; One need not mention that a square is a rectangle as well).
2. Circulate the rows to get $\mathcal{M}_2$ so that i^{th} row of $\mathcal{M}_1$ becomes the $(i+1)^{\text{th}}$ row of $\mathcal{M}_2$, $i = 1, 2, \dots, p-1$. The p^{th} row of $\mathcal{M}_1$ is placed as the first row in $\mathcal{M}_2$.

3. Define $\mathcal{M} = [\mathcal{M}_1 : \mathcal{M}_2]$ (or $[\mathcal{M}_2 : \mathcal{M}_1]$) by appending the two (nearly) magic rectangles side by side.
4. The resulting array $\mathcal{M}$ is a $PBIB[v = pq, b = p, k = 2q, r = 2, (\lambda_1, \lambda_2, \lambda_3) = (2, 1, 0)]$ design.

With $r = 2$ and three associate classes, the number of blocks b must necessarily be greater than 3. Thus we must assume $p > 3$. It can easily be checked that $n_1 = q - 1, n_2 = 2q$, and $n_3 = (p - 3)q$. It can also be verified easily that all the identities for a $PBIB$ design are satisfied and the association scheme and association matrices can also be easily computed. Specifically, treatments listed in i^{th} row of $\mathcal{M}_1$ (equivalently, or of $\mathcal{M}_2$) are first associates ($\lambda_1 = 2$) of each other, treatments in i^{th} row of $\mathcal{M}_1$ and those in i^{th} row of $\mathcal{M}_2$ are second associates ($\lambda_2 = 1$) of each other and finally, those treatments which are neither present in i^{th} row of $\mathcal{M}_1$ nor in the i^{th} row of $\mathcal{M}_2$ are the third associates ($\lambda_3 = 0$) of each other. Informally speaking, treatments in the same row of $\mathcal{M}_1$ are first associates of each other, those in immediately previous row or in the very next row are second associates of treatments in a particular row and the remaining treatments are the third associates of treatments in that particular row.

Further, the three association matrices are given by,

$$P^{(1)} = \begin{bmatrix} q-2 & 0 & 0 \\ 0 & 2q & 0 \\ 0 & 0 & (p-3)q \end{bmatrix},$$

$$P^{(2)} = \begin{bmatrix} 0 & q-1 & 0 \\ q-1 & 0 & q \\ 0 & q & (p-4)q \end{bmatrix} \text{ and}$$

$$P^{(3)} = \begin{bmatrix} 0 & 0 & q-1 \\ 0 & 2q & 0 \\ q-1 & 0 & 0 \end{bmatrix} \text{ if } p=4,$$

$$P^{(3)} = \begin{bmatrix} 0 & 0 & q-1 \\ 0 & q & q \\ q-1 & q & 0 \end{bmatrix} \text{ if } p=5 \text{ and}$$

$$P^{(3)} = \begin{bmatrix} 0 & 0 & q-1 \\ 0 & 0 & 2q \\ q-1 & 2q & (p-6)q \end{bmatrix} \text{ whenever } p \geq 6.$$

From the fact, that in any magic square or in any (nearly) magic rectangle, each treatment appears once and only once, and thus only in one block, such a configuration of treatments does *not* constitute a connected design. This fact has been earlier pointed out in Khattree (2023). However, our designs here are obtained by the appendation of the successive rows (blocks) of these configurations. This makes the corresponding graph of the design matrix a connected one and hence the designs introduced in this article are all connected.

3. THREE ILLUSTRATIVE EXAMPLES

In the present section, the algorithms is illustrated when $(p=4, q=6, \text{ both even})$, $(p=5, q=5, \text{ both odd})$ and when $(p=5, q=4, (\text{odd, even}))$ respectively. The first case corresponds to a magic rectangle, second is about a magic square with p, q both odd and the third case corresponds to a nearly magic rectangle.

A 4×6 Magic Rectangle

$$\text{Let } \mathcal{M}_1 = \begin{bmatrix} 1 & 2 & 3 & 22 & 23 & 24 \\ 19 & 20 & 21 & 4 & 5 & 6 \\ 18 & 17 & 16 & 9 & 8 & 7 \\ 12 & 11 & 10 & 15 & 14 & 13 \end{bmatrix} \text{ and thus}$$

$$\mathcal{M}_2 = \begin{bmatrix} 12 & 11 & 10 & 15 & 14 & 13 \\ 1 & 2 & 3 & 22 & 23 & 24 \\ 19 & 20 & 21 & 4 & 5 & 6 \\ 18 & 17 & 16 & 9 & 8 & 7 \end{bmatrix}.$$

Consequently, the resulting design is obtained by appending the above two side by side as,

$$\mathcal{M}_{(4 \times 12)} = \begin{bmatrix} 1 & 2 & 3 & 22 & 23 & 24 & : & 12 & 11 & 10 & 15 & 14 & 13 \\ 19 & 20 & 21 & 4 & 5 & 6 & : & 1 & 2 & 3 & 22 & 23 & 24 \\ 18 & 17 & 16 & 9 & 8 & 7 & : & 19 & 20 & 21 & 4 & 5 & 6 \\ 12 & 11 & 10 & 15 & 14 & 13 & : & 18 & 17 & 16 & 9 & 8 & 7 \end{bmatrix}.$$

The general association scheme has been described earlier in Section 2. To illustrate, let us consider various associates of Treatment 1 only. These are given below.

Associates $\rightarrow$ of Treatment $\downarrow$	First ($n_1 = 5,$ $\lambda_1 = 2$)	Second ($n_2 =$ $12, \lambda_2 = 1$)	Third ($n_3 = 6,$ $\lambda_3 = 0$)
1	{2, 3, 22, 23, 24}	{12, 11, 10, 15, 14, 13, 19, 20, 21, 4, 5, 6}	{18, 17, 16, 9, 8, 7}

Association schemes for other treatments are self-explanatory from the patterns observed in above. It

is also seen that the corresponding three association matrices in this case are

$$P^{(1)} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 6 \end{bmatrix}, P^{(2)} = \begin{bmatrix} 0 & 5 & 0 \\ 5 & 0 & 6 \\ 0 & 6 & 0 \end{bmatrix} \text{ and}$$

$$P^{(3)} = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 12 & 0 \\ 5 & 0 & 0 \end{bmatrix}.$$

A 5×5 Magic Square

In this case, $\mathcal{M}_1 = \begin{bmatrix} 11 & 24 & 7 & 20 & 3 \\ 4 & 12 & 25 & 8 & 16 \\ 17 & 5 & 13 & 21 & 9 \\ 10 & 18 & 1 & 14 & 22 \\ 23 & 6 & 19 & 2 & 15 \end{bmatrix}$. The

square $\mathcal{M}_2$ is similarly defined by circulating the rows of $\mathcal{M}_1$, resulting in,

$$\mathcal{M}_{(5 \times 10)} = \begin{bmatrix} 23 & 6 & 19 & 2 & 15 & : & 11 & 24 & 7 & 20 & 3 \\ 11 & 24 & 7 & 20 & 3 & : & 4 & 12 & 25 & 8 & 16 \\ 4 & 12 & 25 & 8 & 16 & : & 17 & 5 & 13 & 21 & 9 \\ 17 & 5 & 13 & 21 & 9 & : & 10 & 18 & 1 & 14 & 22 \\ 10 & 18 & 1 & 14 & 22 & : & 23 & 6 & 19 & 2 & 15 \end{bmatrix}.$$

The association scheme can be easily derived as shown in the previous example. The corresponding association matrices are

$$P^{(1)} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}, P^{(2)} = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 5 \\ 0 & 5 & 5 \end{bmatrix} \text{ and}$$

$$P^{(3)} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 5 & 5 \\ 4 & 5 & 0 \end{bmatrix}.$$

A 4×5 Nearly Magic Rectangle

Consider the nearly magic rectangle

$$\mathcal{M}_{1*} = \begin{bmatrix} & & & & & sum \\ & 20 & 16 & 11 & 5 & 1 & 53 \\ & 2 & 6 & 12 & 14 & 19 & 53 \\ & 3 & 7 & 9 & 15 & 18 & 52 \\ & 17 & 13 & 10 & 8 & 4 & 52 \\ sum & 42 & 42 & 42 & 42 & 42 & 210 \end{bmatrix}.$$

The right-most column and bottom-most row indicate the sums of the corresponding rows and columns. Clearly the row sums are 53, 53, 52, 52 respectively and to make these sums alternate, we interchange the second and third rows. This yields,

$$\mathcal{M}_1 = \begin{bmatrix} 20 & 16 & 11 & 5 & 1 \\ 3 & 7 & 9 & 15 & 18 \\ 2 & 6 & 12 & 14 & 19 \\ 17 & 13 & 10 & 8 & 4 \end{bmatrix}.$$

Now we create $\mathcal{M}_2$ as earlier and we append the two side by side to get the

$$\mathcal{M}_{(4 \times 10)} = \begin{bmatrix} 17 & 13 & 10 & 8 & 4 & : & 20 & 16 & 11 & 5 & 1 \\ 20 & 16 & 11 & 5 & 1 & : & 3 & 7 & 9 & 15 & 18 \\ 3 & 7 & 9 & 15 & 18 & : & 2 & 6 & 12 & 14 & 19 \\ 2 & 6 & 12 & 14 & 19 & : & 17 & 13 & 10 & 8 & 4 \end{bmatrix}.$$

It is easy to see that the block sums are all equal namely, 105. It is important to mention that here $p(=4)$ was even while $q(=5)$ was odd and thus rows of the nearly magic rectangle had to be arranged in a way that the row sums alternate. No such amendment is required if P was odd and Q was even as the row sums in $\mathcal{M}_1$ will all be same anyway. The association scheme is, again, self evident. The three association matrices turn out to be

$$P^{(1)} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{bmatrix}, P^{(2)} = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 5 \\ 0 & 5 & 0 \end{bmatrix} \text{ and}$$

$$P^{(3)} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 10 & 0 \\ 4 & 0 & 0 \end{bmatrix}.$$

It may also be pointed out that in all such block designs, the column sums are also (nearly) constant and such designs can be viewed as the two-way (nearly) constant block-sum designs when blocking occurs in two directions. Further, if the structure allows, then the rows of $\mathcal{M}_2$ can further be circulated in the same way to get $\mathcal{M}_3$ to further append to $\mathcal{M}_1$, thereby resulting in another *PBIB* design. Another point regarding the magic squares is that a similar (equivalent but not same) design can be obtained by applying the above algorithm to columns. The two designs, those obtained via rows and via columns, can further be arranged one below another to obtain a $2p \times 2p$ design.

As shown by Khattree (2020), stacking $\mathcal{M}'$, the transpose of a magic square below $\mathcal{M}$ also results in a constant block sum partially balanced incomplete block design with $r = 2$. These are two associate class designs as opposed to three associate class designs considered here. However, the number of blocks in that case is twice as many as opposed to the current work where the number of blocks is still P .

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REFERENCES

- Ravindra Khattree. On construction of constant block-sum partially balanced incomplete block designs. *Communications in Statistics-Theory and Methods* 49(11), 2585-2606, 2020. doi: <https://doi.org/10.1080/03610926.2019.1576895>.
- Ravindra Khattree. Constant and nearly constant block-sum partially balanced incomplete block designs and magic rectangles. *Journal of Statistical Theory and Practice*, 17(1):8, 2023. doi: <https://doi.org/10.1007/s42519-022-00307-2>.
- Kaushal Kumar Yadav, Ankit Kumar Singh, Manoj Varma, Satyam Verma, Prabhat Kumar, and Anita Sarkar. Designing of constant block sum PBIB designs for animal experiments. *International Journal of Veterinary Sciences and Animal Husbandry*, SP-9(4):206–209, 2024. <https://www.veterinarypaper.com/pdf/2024/vol9issue4S/PartD/S-9-4-32-331.pdf>.