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Construction of Almost Unbiased Estimator for Unknown Population Mean using Two Auxiliary Variables

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SUMMARY

In this paper, for estimating finite population mean $\bar{Y}$ of study character Y, an almost unbiased estimator, using two auxiliary variables is proposed. The usual ratio, product and ratio-cum-product estimators are biased. In some applications, biasedness of an estimator is disadvantageous. It is important to reduce it for better accuracy. Applying the procedure of Singh and Singh(1991,1993), in this paper, an almost unbiased estimator using two auxiliary variables is proposed for estimating finite population mean $\bar{Y}$. The proposed estimator is almost unbiased up to first order of approximation. Expression for bias and mean square error (MSE) of the proposed estimator is derived up to the first order of approximation. To verify the theoretical findings an empirical study is carried out using two real data sets. One simulation study is also carried out which demonstrates that the bias of the proposed estimator is almost zero and the minimum MSE is equal to the MSE of the two variable regression estimator.

Keywords: Unbiased estimator; Two auxiliary variables; Ratio-cum-product Estimator; Population mean estimation; Mean square error; Bias.

1. INTRODUCTION

In sampling, it is common practice to improve the precision of estimates by incorporating additional information. This information can be used during survey planning, at the estimation stage, or both. This approach allows researchers to refine their methodologies, reduce errors, and obtain more reliable results. A number of techniques for utilizing auxiliary information are provided in the literature on sampling methods (W.G. Cochran, 1977; M.N. Murthy, 1967). One popular method is to use auxiliary variables at the estimation stage through ratio, product, and regression estimators. Regression estimators is generally more efficient than the ratio and product estimators, except in that case where regression line between study and auxiliary variables passes through origin, in which case their efficiency are nearly identical. Despite this, ratio and product estimators are still frequently used because of their straightforward conceptual appeal and practicality. In order to increase the efficiency of the estimator's, a lot of research has been done on their improvement. Several estimators have been developed that performs better than basic estimators such as the sample mean, ratio, and product estimators when a single auxiliary variable is available. Under optimum conditions, these estimators can even be as effective as the regression estimator. However, in many real-world situations, there are multiple auxiliary variables available, enabling researchers to create estimators that incorporate information from various sources. (I. Olkin, 1958) Introduced a multivariate ratio estimator, combining ratio estimators for different auxiliary variables in a weighted linear form to minimize variance. (M.P. Singh, 2025) Extended this concept to multivariate product estimators, applying a similar combination method. (D.Raj, 1965) Proposed a different approach by considering a linear combination of single-difference estimators, an alternative to ratio-based techniques. A real life example in case of multiple auxiliary variables is in agricultural yield

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estimation. Suppose one agency want to estimate the average yield per hectare in a country. They might use two auxiliary variables. One is rainfall and other is fertilizers usage. Since both are positively correlated with yield, a multivariate ratio or regression estimator could be used to improve yield estimation. Linear combination of ratio estimators, as suggested by [3], or a multivariate regression model would provide a more accurate estimate of the yield.

Several researchers have significantly contributed to this field by refining classical estimation techniques and incorporating auxiliary information more effectively. (M.P. Singh, 1965), (M.P. Singh,1967) introduced the ratio-cum-product estimator, which combines ratio and product estimators for increased efficiency in case of two auxiliary variables. (Bandyopadhaya, 5, 1980) Developed improved ratio and product estimators, offering refinements to traditional methods. (Tracy *et al*., 1996) Proposed an alternative to the ratio-cum-product estimator, improving estimation techniques for sample surveys. (Naik and Gupta, 1991) suggested better estimators that use auxiliary variables to lower variance. (Ceccon and Diana, 1996) Created different approaches to maximize estimator performance in different sampling situations. In order to reduce bias and improve precision, (Abu-Dayyeh *et al*., 2003) introduced improvements to classical estimators. (Kadilar and Cingi, 2005**,** Kadivar and Cingi, 2009) Extended ratio and regression-type estimators to broader sampling frameworks for enhanced efficiency.(Singh *et al*., 2005) Studied the efficiency of a dual to ratio-cum-product estimator, offering further improvements in sample survey methodologies. (Singh *et al*., 2009) Introduced a ratio-cum-product type exponential estimator, refining estimation efficiency. (Singh and Singh, 1993) has given a method of construction of almost unbiased estimator using auxiliary information. (Singh and Singh, 1998) Constructed an almost unbiased estimator in case of ratio and product type estimators in systematic sampling. (Singh *et al*., 2024) Constructed an almost unbiased estimators for population coefficient of variation using auxiliary information.

Motivated by these developments, in this paper a new estimator is constructed using two auxiliary variables that extend the ratio-cum-product estimators. These estimators utilize two auxiliary variables to achieve superior efficiency, reducing variance and improving the reliability of population mean estimation.

Consider a finite population U consisting of N units labeled as $U_1, U_2, U_3, \ldots\ldots\ldots\ldots U_N$ where each unit has a unique identifier ranging from 1 to N. The population is fully identifiable since each unit's label is known. Let y and (x, z) represent the study variable and auxiliary variables, respectively. The values of these variables for a given unit U_i (where i=1,2,…,N) are denoted as y_i, x_i and z_i. Here, x and y are positively correlated, while z and y have a negative correlation. Our primary aim is to estimate the population mean, $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ assuming that the population means $\overline{X}$ and $\overline{Z}$ of the auxiliary variables (x_i, z_i) are known. A simple random sample without replacement (SRSWOR) of size n is drawn from U.

The estimator for estimating unknown population mean $\overline{Y}$ is as follows

$$t_R = \left(\frac{\overline{y}}{\overline{x}}\right)\overline{X} \tag{1}$$

and product estimator is,

$$t_P = \frac{\overline{y}\overline{z}}{\overline{Z}} \tag{2}$$

where $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\overline{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$ are the respective sample means for y, x, and z.

The bias and mean square errors (MSEs) of the estimators t_R and t_P, up to the first order of approximation are, respectively:

$$Bias(t_R) = \overline{Y}\gamma\left[C_x^2 - \rho_{yx}C_yC_x\right] \tag{3}$$

$$Bias(t_P) = \overline{Y}\gamma\rho_{yz}C_yC_z \tag{4}$$

MSE of the estimator t_R, t_P are, respectively:

$$MSE(t_R) = \overline{Y}^2\gamma\left[C_y^2 + C_x^2 - 2\rho_{yx}C_yC_x\right] \tag{5}$$

$$MSE(t_P) = \overline{Y}^2\gamma\left[C_y^2 + C_z^2 + 2\rho_{yz}C_yC_z\right] \tag{6}$$

Here, $\gamma = \frac{1}{n}(1-f)$, $f = \frac{n}{N}$, f is known as sampling fraction, C_y and C_x are the population coefficient of variations of study variable Y and auxiliary variable X respectively and defined as

$C_y = \frac{S_y}{\overline{Y}}$ and $C_x = \frac{S_x}{\overline{X}}$. ρ is the correlation coefficient between X and Y.

$S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_i - \overline{Y})^2$ is the population mean square of the study variable Y,

$S_x^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(X_i - \overline{X})^2$ is the population mean square of the auxiliary variable X,

$S_z^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(Z_i - \overline{Z})^2$ is the population mean square of the auxiliary variable Z, $S_{xy} = \frac{1}{(N-1)}\sum_{i=1}^{N}(X_i - \overline{X})(Y_i - \overline{Y})$ is the population covariance of the study and auxiliary variable Y and X,

$S_{yz} = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_i - \overline{Y})(Z_i - \overline{Z})$, is the population covariance of the study and auxiliary variable Y and Z,

$S_{xz} = \frac{1}{(N-1)}\sum_{i=1}^{N}(X_i - \overline{X})(Z_i - \overline{Z})$, is the population covariance of the auxiliary variables X and Z.

$s_y^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(y_i - \overline{y})^2$ - is the sample mean square of the study variable y,

$s_x^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(x_i - \overline{x})^2$ - is the sample mean square of the auxiliary variable x.

Let,

$$e_0 = \frac{\overline{y} - \overline{Y}}{\overline{Y}},\ e_1 = \frac{\overline{x} - \overline{X}}{\overline{X}},\ e_2 = \frac{(\overline{z} - \overline{Z})}{\overline{Z}},$$

$$\overline{y} = \overline{Y}(1+e_0),\ \overline{x} = \overline{X}(1+e_1),\ \overline{z} = \overline{Z}(1+e_2)$$

$$E(e_0) = E(e_1) = E(e_2) = 0$$

$$E(e_0^2) = \gamma C_y^2,\ E(e_1^2) = \gamma C_x^2,\ E(e_2^2) = \gamma C_z^2,$$

(M.P. Singh, 1967) improved the ratio and product methods of estimation and suggested the 'ratio-cum-product' estimator for $\overline{Y}_{RP}$ as :

$$t_{RP} = \overline{y}\frac{\overline{X}}{\overline{x}}\frac{\overline{z}}{\overline{Z}} \tag{7}$$

(Koyuncu and Kadilar, 2009) suggested different ratio-cum-product estimators of population mean.

t_{RR} is the ratio-cum-ratio estimator and is defined as:

$$t_{RR} = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)\left(\frac{\overline{Z}}{\overline{z}}\right) \tag{8}$$

t_{PP} is the product-cum-product estimator and is defined as:

$$t_{PP} = \overline{y}\left(\frac{\overline{x}}{\overline{X}}\right)\left(\frac{\overline{z}}{\overline{Z}}\right) \tag{9}$$

The bias and MSE of the estimator t_{RR} is obtained as:

$$Bias(t_{RR}) = \overline{Y}\gamma\left[C_y^2 + C_z^2 - \rho_{yx}C_yC_x - \rho_{yz}C_yC_z + \rho_{xz}C_xC_z\right] \tag{10}$$

$$MSE(t_{RR}) = \overline{Y}^2\gamma\left[C_y^2 + C_x^2 + C_z^2 - 2\rho_{yx}C_yC_x - 2\rho_{yz}C_yC_z + 2\rho_{xz}C_xC_z\right] \tag{11}$$

The bias and MSE of the estimator t_{PP} is obtained as:

$$Bias(t_{PP}) = \overline{Y}\gamma\left[\rho_{yx}C_yC_x + \rho_{yz}C_yC_z + \rho_{xz}C_xC_z\right] \tag{12}$$

$$MSE(t_{PP}) = \overline{Y}^2\gamma\left[C_y^2 + C_x^2 + C_z^2 + 2\rho_{yx}C_yC_x + 2\rho_{yz}C_yC_z + 2\rho_{xz}C_xC_z\right] \tag{13}$$

2. PROPOSED ALMOST UNBIASED ESTIMATOR

In this section, an almost unbiased estimator t_{h1} for estimating population mean using two auxiliary variables has been proposed. To achieve this, following three estimators m_0, m_1 and m_2 are considered as:

$$\mathrm{m}_0 = \overline{y} \tag{14}$$

Bias of the estimator m_0 is

$$Bias(m_0) = 0 \tag{15}$$

Variance of the estimator m_0 is

$$Var(m_0) = \overline{Y}^2\gamma C_y^2 \tag{16}$$

Ratio cum product estimator m_1 is written as

$$m_1 = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\left(\frac{\bar{z}}{\bar{Z}}\right) \tag{17}$$

Bias of the estimator m_1 is

$$Bias(\mathrm{m}_1) = \bar{Y}\gamma\left[C_x^2 - \rho_{yx}C_yC_x - \rho_{yz}C_yC_z + \rho_{xz}C_xC_z\right] \tag{18}$$

MSE of the estimator m_1 is

$$MSE(\mathrm{m}_1) = \bar{Y}^2\gamma\left[C_y^2 + C_x^2 + C_z^2 - 2\rho_{yx}C_yC_x + 2\rho_{yz}C_yC_z - 2\rho_{xz}C_xC_z\right] \tag{19}$$

Exponential ratio product estimator m_2 is written in equation (39) as:

$$m_2 = \bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\exp\left(\frac{\bar{z}-\bar{Z}}{\bar{z}+\bar{Z}}\right) \tag{20}$$

Bias of the estimator of m_2 is

$$Bias(\mathrm{m}_2) = \bar{Y}\gamma\left[\frac{3}{8}*C_x^2 - \frac{1}{4}*C_z^2 - \frac{1}{4}\rho_{yx}C_yC_x + \frac{1}{2}\rho_{yz}C_yC_z - \frac{1}{4}\rho_{xz}C_xC_z\right] \tag{21}$$

MSE of the estimator m_2 is

$$MSE(\mathrm{m}_2) = \bar{Y}^2\gamma\left[C_y^2 + \frac{1}{4}C_x^2 + \frac{1}{4}C_z^2 - 2\rho_{yx}C_yC_x + 2\rho_{yz}C_yC_z - \frac{1}{2}\rho_{xz}C_xC_z\right] \tag{22}$$

Let, $m_0 = \bar{y}$, $m_1 = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\left(\frac{\bar{z}}{\bar{Z}}\right)$ and $m_2 = \bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\exp\left(\frac{\bar{z}-\bar{Z}}{\bar{z}+\bar{Z}}\right)$ are the three estimators, such that m_0, m_1 and $m_2 \in L$, where L is the set of all possible estimators for estimating the population mean.

Table 1. Members of the proposed family t_{h1} of estimators

l_0	l_1	l_2	**Estimators**
1	0	0	$\bar{y}$
0	1	0	$\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\left(\frac{\bar{z}}{\bar{Z}}\right)$
0	0	1	$\bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\exp\left(\frac{\bar{z}-\bar{Z}}{\bar{z}+\bar{Z}}\right)$

Following approach of Singh and Singh (1991, 1993), an almost unbiased estimator t_{h1} is defined as:

$$t_{h1} = \sum_{i=0}^{2} l_i m_i \in L \tag{23}$$

$$t_{h1} = l_0\bar{y} + l_1\bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)\left(\frac{\bar{z}}{\bar{Z}}\right) + l_2\bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right)\exp\left(\frac{\bar{z}-\bar{Z}}{\bar{z}+\bar{Z}}\right) \tag{24}$$

$$\text{For } \sum_{i=0}^{2} l_i = 1,\ l_i \in \mathrm{R} \tag{25}$$

where, $l_i(i = 0,1,2)$ denotes the statistical constants and R denotes the set of real numbers.

To determine the bias and MSE of the estimator t_{h1}, express the estimator t_{h1} in terms of the error as:

$$t_{h1} = \bar{Y}\left(1+e_0\right)\left[l_0 + l_1\left(1+e_1\right)^{-1}\left(1+e_2\right) + l_2\exp\left(-e_1\left(2+e_1\right)^{-1}\right)\exp\left(e_2\left(2+e_2\right)^{-1}\right)\right] \tag{26}$$

Equation (26) can be re-written as follows:

$$t_{h1} = \bar{Y}\begin{bmatrix} 1 + e_0 - \left(\alpha_1 + \frac{\alpha_2}{2}\right)e_1 + \left(\alpha_1 + \frac{\alpha_2}{2}\right)e_2 - \\ \left(\alpha_1 + \frac{\alpha_2}{2}\right)e_0e_1 + \left(\alpha_1 + \frac{\alpha_2}{2}\right)e_0e_2 - \\ \left(\alpha_1 + \frac{1}{4}\alpha_2\right)e_1e_2 + \left(\alpha_1 + \frac{3}{8}\alpha_2\right)e_1^2 - \frac{1}{4}\alpha_2 e_2^2 \end{bmatrix} \tag{27}$$

Subtracting $\bar{Y}$ and then taking expectation of both the sides the bias of the estimator t_{h1}, up to the first order of approximation is written as:

$$Bias(t_{h1}) = \bar{Y}\gamma\begin{bmatrix} H_1C_x^2 + H_1\rho_{yz}C_yC_z + \frac{1}{4}l_2\rho_{xz}C_xC_z - \\ H_1\rho_{yx}C_yC_x - H_1\rho_{xz}C_xC_z - \frac{1}{8}l_2C_x^2 - \\ \frac{1}{4}l_2C_z^2 \end{bmatrix} \tag{28}$$

$$H_1 = (\alpha_1 + \frac{1}{2}\alpha_2) \tag{29}$$

Equation (27) can be written as follows:

$$\left(t_{h1}-\overline{Y}\right)\simeq\left[e_0+H_1e_2-H_1e_1\right] \tag{30}$$

Squaring both sides of equation(30)and then taking expectation, MSE of the estimator t_{h1}, up to the first order of approximation, is written as:

$$MSE(t_{h1})=\overline{Y}^2\gamma\begin{bmatrix}C_y^2+H_1^2C_x^2+H_1^2C_z^2-2H_1\rho_{yx}C_yC_x+\\2H_1\rho_{yz}C_yC_z-2H_1^2\rho_{xz}C_xC_z\end{bmatrix} \tag{31}$$

Which is minimum when

$$H_{1opt}=\frac{(\rho_{yx}C_yC_x-\rho_{yz}C_yC_z)}{\left(C_x^2+C_z^2-2\rho_{xz}C_xC_z\right)} \tag{32}$$

Putting this minimum value of H_{1opt} in equation (31), $Min.MSE$ of the estimator t_{h1} is obtained as follows:

$$Min.MSE(t_{h1})=\overline{Y}^2\gamma\begin{bmatrix}C_y^2+H_1^2C_x^2+H_1^2C_z^2-\\2H_1\rho_{yx}C_yC_x+2H_1\rho_{yz}C_yC_z-\\2H_1^2\rho_{xz}C_xC_z\end{bmatrix} \tag{33}$$

From equation (29) and (32), the value of H_{1opt} is written as follows

$$H_{1opt}=\left(\alpha_1+\frac{\alpha_2}{2}\right)=\frac{(\rho_{yx}C_yC_x-\rho_{yz}C_yC_z)}{\left(C_x^2+C_z^2-2\rho_{xz}C_xC_z\right)} \tag{34}$$

From equation (24) and (29), there are only two equations with three unknowns. It is not possible to find unique values for $l_i's$ $(i=0,1,2)$. To get the values of $l_i's$ a linear restrictions is imposed as:

$$\sum_{i=0}^{2}l_iB(\mathrm{m}_i)=0 \tag{35}$$

$$l_0B(\mathrm{m}_0)+l_1B(\mathrm{m}_1)+l_2B(\mathrm{m}_2)=0 \tag{36}$$

Where $B(\mathrm{m}_i)$ denotes the bias of the i^{th} estimator.

Equation can be written in the matrix from as:

$$\begin{bmatrix}1&1&1\\0&1&\frac{1}{2}\\0&B(\mathrm{m}_1)&B(\mathrm{m}_2)\end{bmatrix}\begin{bmatrix}l_0\\l_1\\l_2\end{bmatrix}=\begin{bmatrix}1\\H_1\\0\end{bmatrix} \tag{37}$$

From the system of equation (37), the unique values of the $l_i's$ are obtained as:

$$l_0=\frac{B(\mathrm{m}_2)-\frac{1}{2}B(\mathrm{m}_1)-H_1B(\mathrm{m}_2)+H_1B(\mathrm{m}_1)}{B(\mathrm{m}_2)-\frac{1}{2}B(\mathrm{m}_1)} \tag{38}$$

$$l_1=\frac{H_1B(\mathrm{m}_2)}{B(\mathrm{m}_2)-\frac{1}{2}B(\mathrm{m}_1)} \tag{39}$$

$$l_2=\frac{-H_1B(\mathrm{m}_1)}{B(\mathrm{m}_2)-\frac{1}{2}B(\mathrm{m}_1)} \tag{40}$$

Such that

$$l_0+l_1+l_2=1 \tag{41}$$

Use of these $l_i's$ $(i=0,1,2)$ remove the bias up to terms of order $O(n^{-1})$

3. THEORETICAL COMPARISON

In this section the efficiency of the proposed estimator t_h is compared with all other existing estimator considered here

1. Estimator t_{h1} is more efficient than t_0 if

$$Min.MSE(t_{h1})<\mathrm{MSE(t_0)}$$

$$Min.MSE(t_1)<\overline{Y}^2\gamma C_y^2 \tag{42}$$

2. Estimator t_{h1} is more efficient than t_{RR} if

$$Min.MSE(t_{h1})<\mathrm{MSE(t}_{RR})$$

$$Min.MSE(t_{h1})<\overline{Y}^2\gamma\begin{bmatrix}C_y^2+C_x^2+C_z^2-2\rho_{yx}C_yC_x-\\2\rho_{yz}C_yC_z+2\rho_{xz}C_xC_z\end{bmatrix} \tag{43}$$

3. Estimator t_{h1} is more efficient than t_{PP} if

$$Min.MSE(t_{h1})<\mathrm{MSE(t}_{PP})$$

$$Min.MSE(t_{h1})<\overline{Y}^2\gamma\begin{bmatrix}C_y^2+C_x^2+C_z^2+2\rho_{yx}C_yC_x+\\2\rho_{yz}C_yC_z+2\rho_{xz}C_xC_z\end{bmatrix} \tag{44}$$

4. Estimator t_{h1} is more efficient than m_1 if

$$Min.MSE(t_{h1})<\mathrm{MSE(m_1)}$$

$$Min.MSE(t_{h1})<\overline{Y}^2\gamma\begin{bmatrix}C_y^2+C_x^2+C_z^2-2\rho_{yx}C_yC_x+\\2\rho_{yz}C_yC_z-2\rho_{xz}C_xC_z\end{bmatrix} \tag{45}$$

5. Estimator t_{h1} is more efficient than m_2 if

$$Min.MSE(t_{h1}) < \mathrm{MSE}(\mathrm{m}_2)$$

$$Min.MSE(t_{h1}) < \bar{Y}^2\gamma\left[\begin{array}{l} C_y^2+\frac{1}{4}C_x^2+\frac{1}{4}C_z^2-2\rho_{yx}C_yC_x+ \\ 2\rho_{yz}C_yC_z-\frac{1}{2}\rho_{xz}C_xC_z \end{array}\right] \tag{46}$$

4. EMPIRICAL AND SIMULATION STUDY

In this section, the performance of the constructed estimators over various other estimators are illustrated through two real data sets.

Population 1. (M.P. Singh, 1969)

y: The eggs produced in 1990,

x: The price per dozen in 1990,

z: The price per dozen in 1999,

$\bar{Y}$ = 1357.622, $\bar{X}$ = 78.29, $\bar{Z}$ = 75.872, C_y^2 = 1.497297, C_x^2 = 0.0741410, C_z^2 = 0.085463, ρ_{yx} = -0.28883, ρ_{yz} = -0.302228 , ρ_{xz} = 0.987327, N=50 , n=15.

Population 2 (M.P. Singh, 1969)

y: Number of females employed

x: Number of females in service

z: Number of educated females.

$\bar{Y}$ = 7.46, $\bar{X}$ = 5.31, $\bar{Z}$ = 179, C_y^2 = 0.5046, C_x^2 = 0.5737, C_z^2 = 0.0633, ρ_{yx} =0.7737, ρ_{yz} = -0.2070, ρ_{xz} = -0.0033 , N= 61 , n=20.

Table 2. Values of $l_i(i=0,1,2)$

S.N.	Scalars	Population 1	Population 2
1	l_0	-7.8772	2.9146
2	l_1	0.9416	3.3350
3	l_2	7.9356	-5.2496
4	$l_1+\left(\frac{1}{2}\right)l_2=H_1$	4.9094	0.7102

Using these values of $l_i(i=0,1,2)$ given in the table , one can reduce the bias up to the order $\mathrm{O(n^{-1})}$ in estimator t_{h1} .

Table 3. Bias of the existing and proposed estimators

Estimators	Population 1 (Bias)	Population 2 (Bias)
t_0	0.0	0.0
t_{PP}	-0.0010	0.0949
t_{RR}	0.0149	0.0471
t_{RP}	0.0029	-0.0310
$t_{h1(Min.)}$	0.00	0.0

Table 4. The MSE and PRE of the existing and proposed estimators in case of population 1

Estimators	MSE	PRE
t_0	0.0120	100
t_{PP}	0.0112	106.540
t_{RR}	0.0178	67.361
t_{RP}	0.0118	101.446
$t_{h1(Min.)}$	0.0115	104.053

Table 5. The MSE and PRE of the existing and proposed estimators in case of population 2

Estimator	MSE	PRE
t_0	0.944	100
t_{PP}	3.551	26.573
t_{RR}	0.714	132.175
t_{RP}	0.442	213.541
$t_{h1(Min.)}$	0.342	276.205

4.1 Simulation Study

A simulation study is carried out to check the proposed estimator's relative efficiency (RE) with the existing estimators. This is done via the following steps

1. 1000 observationsare generated from a tri-variate normal distribution with mean $(\mu_x,\mu_y\ ,\ \mu_z)=(200,150,300)$ and covariance matrix $\begin{bmatrix} \sigma_y^2 & \rho_{yx}\sigma_y\sigma_x & \rho_{yz}\sigma_y\sigma_z \\ \rho_{yx}\sigma_y\sigma_x & \sigma_x^2 & \rho_{xz}\sigma_x\sigma_z \\ \rho_{yz}\sigma_y\sigma_z & \rho_{xz}\sigma_x\sigma_z & \sigma_z^2 \end{bmatrix}$,

where, $\sigma_y^2 = 841$,$\sigma_x^2 = 625$, $\sigma_z^2 = 961$ and the correlation coefficients are $\rho_{yx} = 0.8, \rho_{yz} = -0.9, \rho_{xz} = -0.7$.

2. A Sample of size 75, 100 and 150 is taken randomly from N=1000.
3. Use the sample data from (2) to obtain the value of constants, Bias, MSE of all the estimators under study.
4. The whole process of the simulation was replicated 10000 times to obtain MSEs, the average of the 10000 values obtained are the MSE of each estimator of population mean.
5. The Percentage Relative Efficiency (PRE) of the estimator over the mean per unit estimator is as follows,

$$PRE(Estimator) = \frac{MSE(t_0)}{MSE(estimator)} * 100$$

Table 6. Values of $l_i (i = 0,1,2)$

Scalars	Values (n=75)	Scalars (n=100)	Values (n=150)
l_0	-0.02083	-0.01499	-0.00876
l_1	0.00371	0.00934	0.01565
l_2	1.01713	1.00565	0.99311
$l_1 + \left(\frac{1}{2}\right) l_2 = H_1$	0.51227	0.51217	0.5122

Using these values of $l_i (i = 0,1,2)$ given in the table , one can reduce the bias up to the order $O(n^{-1})$ in estimator.

5. RESULT AND DISCUSSION

Tables 4 and 5 compares estimators according to their PRE and MSE. In Table 4 the baseline estimator t_0, which is the reference for PRE, set at 100, has the MSE of 0.0120. In case of Population 1 Population 2 from the Table 4 and 5the most accurate and effective estimator is $t_{h1(Min.)}$ -which performs the best with the lowest MSE of 0.342 and PRE of 276.205.The Table 2 provides scalar values that indicate how the proposed estimator reduces its bias for two different populations. These scalars, l_0, l_1 and l_2 scalars are adjustment factors that reduces bias of proposed estimator t_{h1}. The Table 3 represents the bias values of different estimators for two populations. The simulation study evaluated estimator performance by generating a tri-variate normal population (N=1000) with specified means and covariance structure. Random samples of sizes n=75, 100, and 150 are drawn, with the process repeated 10,000 times to compute average MSE and PRE values. The results shows that the proposed estimator $t_{h1(Min.)}$ consistently achieved zero bias and better efficiency across all the sample sizes with PRE values 552 (n=75), 550(n=100) and 547(n=150) significantly performing better comparison then other estimators.

6. CONCLUSION

In this paper, an almost unbiased estimator t_{h1} has been proposed. From the MSE and PRE values, it was observed that the proposed estimator t_{h1} performs better than all other existing estimators considered in this paper. Scalar values, in Table 2 are mentioned that reduces the bias up to the first order of approximation. So, for practical purposes when using almost unbiased estimator the estimator should be preferred.

Table 7. The MSE and PRE of the existing and proposed estimators in case of simulation study

Estimators	N=1000, n=75			N=1000, n=100			N=1000, n=150		
	Bias	MSE	PRE	Bias	MSE	PRE	Bias	MSE	PRE
t_0	0.00	10.8062	100	0.00	8.0880	100	0.00	5.3539	100
t_{PP}	0.0331	12.1909	89.00	0.0247	9.1018	89.00	0.0165	6.0391	89.00
t_{RR}	-0.0153	25.4241	43.00	-0.0115	19.0329	42.00	-0.0076	12.5864	43.00
t_{RP}	0.0273	9.9945	108.00	0.0204	7.4792	108.00	0.0135	4.9496	108.00
$t_{h1(Min.)}$	0.00	1.9586	552.00	0.00	1.4698	550.00	0.00	0.9787	547.00

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