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Rotatable Mixed-Level Response Surface Designs Incorporating Neighbour Effects

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SUMMARY

In this paper, response surface designs have been constructed for a specific factor level combination that ensures the rotatability of the design. Here, the underlying situation is considered when experimental units are suspected of having neighbouring or overlap effects from left and right adjacent units. Furthermore, conditions have been established to ensure orthogonal estimation of the model parameters, thereby ensuring the constancy of the variance of the estimated response. A new methodological approach to the construction of designs of the form $s_1^{n_1} \times s_2^{n_2} \times s_3^{n_3}$ and a particular case satisfying derived conditions has been developed. R functions are also developed to generate the proposed series of designs.

Keywords: Overlap effect; response surface methodology; rotatable design; R-package

Tribute to Professor M. N. Das

Our research in the area of Design of Experiments has been profoundly influenced by the pioneering contributions of Professor M. N. Das. His authoritative textbook *Design and Analysis of Experiments* (co-authored with Prof. N. C. Giri) and his seminal research papers, particularly on response surface designs, continue to serve as enduring references for researchers and practitioners alike. Professor Das was a teacher to many of our teachers, and through them, his intellectual legacy, scholarly rigor, and deep insight into experimental design have inspired successive generations of statisticians. His clarity of thought and dedication to advancing the theory and application of experimental designs have left an indelible mark on the field of agricultural statistics. We gratefully acknowledge the lasting influence of his work on our academic and research endeavours.

1. INTRODUCTION

Response Surface Methodology (RSM), introduced by Box and Wilson (1951), is a strong statistical technique used to model and analyze systems where multiple input factors influence a response variable. The primary goal of RSM [Box and Draper (1987), Khuri and Cornell (1996), Myers (1995), Myers *et al.* (2009), Das *et al.* (1999) and Hemavathi *et al.* (2021)] is to identify the optimal settings of these factors to achieve a desired response. This is accomplished systematically.

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Suppose $x_1, x_2, \ldots, x_v$ are independently distributed variables influencing the response variable y, and the experiment was conducted in N runs. In this case, the response function can be approximated by a polynomial model shown below:

$$y_j = f(x_{1j}, x_{2j}, \ldots, x_{vj}) + e_j,$$

where y_j is the response from the j^{th} treatment combination (j = 1, 2, ..., N) and the level of the i^{th} factor (i = 1, 2, ..., v) in the j^{th} combination represented as x_{ij}. The function $f(\cdot)$ represents the functional relationship between the response and the input variables. And e_j is the random error associated with y_j follows as $N(0, \sigma^2)$.

In RSM, generally neighbouring effects are not considered, but in some practical situations it is found that neighbour effects can play a vital role in drastically reducing experimental errors, and the response is predicted with more precision [Jaggi *et al*. (2010); Bartlett (1938, 1978); Dalal *et al*. (2022); Draper and Guttman (1980); Papadakis (1937)].

For example, a spray of pesticides may fall on the neighbour plots due to wind, or higher root growth of one plant may cause a nutrient deficiency in the plants in adjacent plots. Hence, it is reasonable to assume that the response observed from a particular plot is affected not only by the treatment combination applied to it but also by those applied to the neighbouring plots. For more details, one can refer to Sarika *et al.* (2009, 2013); Varghese *et al.* (2013, 2016, 2019, 2020), Kumar *et al.* (2020), Verma *et al.* (2021), and Dalal *et al.* (2025). Here, a general approach is presented for constructing response surface designs of the form $s_1^{n_1} \times s_2^{n_2} \times s_3$ has been developed when $n_3 = 1$, incorporating neighbour effects.

2. MATERIALS AND METHODS

The methodology for constructing an asymmetrical response surface in the presence of neighbour effects from adjoining units is described below. It is assumed that the neighbouring experimental units exert equal effects.

The response model that accounts for the effects of the immediate left and right neighbouring units is defined as follows:

$$y_{j'} = \sum_{j, j'=1}^{N} g_{jj'} f(x_j) + e_{j'}, \quad j = 1, 2, ..., N \tag{2.1}$$

where

$$g_{jj'} = \begin{cases} 1 & , \text{if } j = j' \\ \alpha & , |\alpha| < 1, \text{if } |j - j'| = 1 \text{ i.e. units are physically adjacent} \\ 0 & , \text{otherwise} \end{cases} \tag{2.2}$$

Here, $\alpha (0 \le \alpha \le 1)$ represents the effect coming from the adjacent left and right neighbouring units. $f(x_j)$ is the function by which the response and the input variables are related.

It should be noted that the experimental design layout for estimating this model includes two additional border units at each end. No observations are recorded from these border units; therefore, they are not included in the model.

Model (2.1) can be expressed equivalently as

$$\mathbf{Y} = \mathbf{GX\beta} + \mathbf{e},$$

where $\mathbf{G} = (g_{jj'})$ is the symmetric neighbour matrix of order $N \times (N + 2)$, $\mathbf{X}$ is an $(N + 2) \times p$ matrix containing the N design points (runs) along with two additional border units (the treatment combinations applied to these border units are the same as those from the N design points), p is the number of parameters, $\boldsymbol{\beta}$ is a vector of parameters to be estimated of order $p \times 1$, and $\mathbf{e}$ is a vector of errors of order $N \times 1$, which follows $N(0, \sigma^2\mathbf{I})$.

When $\mathbf{G}$ is known, the ordinary least squares (OLS) estimate of $\boldsymbol{\beta}$, in the presence of neighbour effects, is given by

$$\hat{\boldsymbol{\beta}} = (\mathbf{Z'Z})^{-1}\mathbf{Z'}Y,$$

where, $\mathbf{Z} = \mathbf{GX}$ with $D(\hat{\boldsymbol{\beta}}) = \sigma^2 (\mathbf{Z'Z})^{-1}$.

If we consider an asymmetrical response surface with factors n_1, n_2 and, n_3 each at s_1, s_2 and, s_3 levels, respectively, resulting in $s_1^{n_1} \times s_2^{n_2} \times s_3^{n_3}$ combinations. The form of $f(x_j)$ in (2.1) considered here for this setup is as follows:

$$\begin{aligned} f(x_j) = \beta_0 &+ \sum_{i=1}^{n_1} \beta_i x_i + \sum_{i=1}^{n_1} \beta_{ii} x_i^2 + ... + \sum_{i=1}^{n_1} \beta_{ii..i} x_i^{s_1-1} + \\ &\sum_{i=n_1+1}^{n_1+n_2} \beta_i x_i + \sum_{i=n_1+1}^{n_1+n_2} \beta_{ii} x_i^2 + ... + \sum_{i=n_1+1}^{n_1+n_2} \beta_{ii..i} x_i^{s_2-1} + \\ &\sum_{i=n_1+n_2+1}^{n_1+n_2+n_3} \beta_i x_i + \sum_{i=n_1+n_2+1}^{n_1+n_2+n_3} \beta_{ii} x_i^2 + ... + \sum_{i=n_1+n_2+1}^{n_1+n_2+n_3} \beta_{ii..i} x_i^{s_3-1} \end{aligned} \tag{2.3}$$

where β_0, β_i's [associated with the linear terms of $v = n_1 + n_2 + n_3$ factors], β_{ii} [associated with the quadratic term of $n_1 + n_2 + n_3$ factors], and $\beta_{ii...i}$ [associated with $(s_i\text{-}1)^{th}$ order term of n_i^{th} factor] are parameters to be estimated. The total number of parameters to be estimated in the above model is $p = n_1(s_1-1) + n_2(s_2-1) + n_3(s_3-1) + 1$.

2.1 Response Surface Methodology for $2^2 \times 3^2 \times 4$ Factorial with Neighbour Effects

Consider $n_1 = n_2 = 2$, $s_1 = 2$, $s_2 = 3$ and $s_3 = 4$ i.e. two factors at levels two, three and one factor with levels four, respectively.

The form of $f(x_j)$ is as follows:

$$f(x_j) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4 + \beta_5 x_5 + \beta_{33} x_3^2 + \beta_{44} x_4^2 + \beta_{55} x_5^2 + \beta_{555} x_5^3$$

The matrix **X** of order (N+2) × 10 with two border lines is written as

$$\mathbf{X}_{(N+2)\times 10} = \begin{bmatrix} 1 & x_{1N} & x_{2N} & x_{3N} & x_{4N} & x_{5N} & x_{3N}^2 & x_{4N}^2 & x_{5N}^2 & x_{5N}^3 \\ 1 & x_{11} & x_{21} & x_{31} & x_{41} & x_{51} & x_{31}^2 & x_{41}^2 & x_{51}^2 & x_{51}^3 \\ 1 & x_{12} & x_{22} & x_{32} & x_{42} & x_{52} & x_{32}^2 & x_{42}^2 & x_{52}^2 & x_{52}^3 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 1 & x_{1i} & x_{2i} & x_{3i} & x_{4i} & x_{5i} & x_{3i}^2 & x_{4i}^2 & x_{5i}^2 & x_{5i}^3 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 1 & x_{1N} & x_{2N} & x_{3N} & x_{4N} & x_{5N} & x_{3N}^2 & x_{4N}^2 & x_{5N}^2 & x_{5N}^3 \\ 1 & x_{11} & x_{21} & x_{31} & x_{41} & x_{51} & x_{31}^2 & x_{41}^2 & x_{51}^2 & x_{51}^3 \end{bmatrix}$$

The structure of the **G** matrix is

$$\mathbf{G}_{N\times(N+2)} = \begin{bmatrix} \alpha & 1 & \alpha & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & \alpha & 1 & \alpha & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & \alpha & 1 & \alpha & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & \alpha & 1 & \alpha & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & & & . & . \\ . & . & . & . & . & . & & . & & . & . \\ . & . & . & . & . & . & & & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 & \alpha \end{bmatrix}$$

Further,

$$\mathbf{Z}_{N\times 10} = \mathbf{GX} = [\boldsymbol{\beta}\ \mathbf{A}_1\ \mathbf{A}_2\ \mathbf{A}_3\ \mathbf{A}_4\ \mathbf{A}_5\ \mathbf{B}_3\ \mathbf{B}_4\ \mathbf{B}_5\ \mathbf{C}_5]$$

where, $\boldsymbol{\beta}$ is vector of N x 1 which consists of (1+2 α) (say, θ), $\mathbf{A}_i$ is a vector of order N × 1 that contains $A_{ij} = x_{ij} + \alpha[x_{i(j-1)\bmod(N)} + x_{i(j+1)\bmod(N)}]$, $\mathbf{B}_i$ is a vector of order N × 1 that contains $B_{ij} = x_{ij}^2 + \alpha[x_{i(j-1)\bmod(N)}^2 + x_{i(j+1)\bmod(N)}^2]$ and $\mathbf{C}_i$ is a vector of order N × 1 that contains $C_{5j} = x_{ij}^3 + \alpha[x_{i(j-1)\bmod(N)}^3 + x_{i(j+1)\bmod(N)}^3]$ $(i = 1,2,...,5;\ j = 1,2,...,N)$

$$\mathbf{Z'Z} = \begin{bmatrix} \theta^2 N & \theta\sum_{j=1}^N A_{1j} & \theta\sum_{j=1}^N A_{2j} & \theta\sum_{j=1}^N A_{3j} & \theta\sum_{j=1}^N A_{4j} & \theta\sum_{j=1}^N A_{5j} & \theta\sum_{j=1}^N B_{3j} & \theta\sum_{j=1}^N B_{4j} & \theta\sum_{j=1}^N B_{5j} & \theta\sum_{j=1}^N C_{5j} \\ & \sum_{j=1}^N A_{1j}^2 & \sum_{j=1}^N A_{1j}A_{2j} & \sum_{j=1}^N A_{1j}A_{3j} & \sum_{j=1}^N A_{1j}A_{4j} & \sum_{j=1}^N A_{1j}A_{5j} & \sum_{j=1}^N A_{1j}B_{3j} & \sum_{j=1}^N A_{1j}B_{4j} & \sum_{j=1}^N A_{1j}B_{5j} & \sum_{j=1}^N A_{1j}C_{5j} \\ & & \sum_{j=1}^N A_{2j}^2 & \sum_{j=1}^N A_{2j}A_{3j} & \sum_{j=1}^N A_{2j}A_{4j} & \sum_{j=1}^N A_{2j}A_{5j} & \sum_{j=1}^N A_{2j}B_{3j} & \sum_{j=1}^N A_{2j}B_{4j} & \sum_{j=1}^N A_{2j}B_{5j} & \sum_{j=1}^N A_{2j}C_{5j} \\ & & & \sum_{j=1}^N A_{3j}^2 & \sum_{j=1}^N A_{3j}A_{4j} & \sum_{j=1}^N A_{3j}A_{5j} & \sum_{j=1}^N A_{3j}B_{3j} & \sum_{j=1}^N A_{3j}B_{4j} & \sum_{j=1}^N A_{3j}B_{5j} & \sum_{j=1}^N A_{3j}C_{5j} \\ & & & & \sum_{j=1}^N A_{4j}^2 & \sum_{j=1}^N A_{4j}A_{5j} & \sum_{j=1}^N A_{4j}B_{3j} & \sum_{j=1}^N A_{4j}B_{4j} & \sum_{j=1}^N A_{4j}B_{5j} & \sum_{j=1}^N A_{4j}C_{5j} \\ & & & & & \sum_{j=1}^N A_{5j}^2 & \sum_{j=1}^N A_{5j}B_{3j} & \sum_{j=1}^N A_{5j}B_{4j} & \sum_{j=1}^N A_{5j}B_{5j} & \sum_{j=1}^N A_{5j}C_{5j} \\ & & & & & & \sum_{j=1}^N B_{3j}^2 & \sum_{j=1}^N B_{3j}B_{4j} & \sum_{j=1}^N B_{3j}B_{5j} & \sum_{j=1}^N B_{3j}C_{5j} \\ & & & & & & & \sum_{j=1}^N B_{4j}^2 & \sum_{j=1}^N B_{4j}B_{5j} & \sum_{j=1}^N B_{4j}C_{5j} \\ & & & & & & & & \sum_{j=1}^N B_{5j}^2 & \sum_{j=1}^N B_{5j}C_{5j} \\ & & & & & & & & & \sum_{j=1}^N C_{5j}^2 \end{bmatrix}$$

The following conditions are necessary to ensure near orthogonal estimation of parameters:

a. $\sum_{u=1}^{N}\prod_{i=1}^{3} x_{iu}^{\omega_i} = 0$ for ω_i =0, 1, 2, or 3 and $\sum \omega_i < 6$

b. $\sum_{u=1}^{N} x_{1u}^2 = \delta_1$, $\sum_{u=1}^{N} x_{2u}^2 = \delta_2$, $\sum_{u=1}^{N} x_{3u}^2 = \delta_3$

c. $\sum_{u=1}^{N} x_{2u}^4 = \gamma_1$, $\sum_{u=1}^{N} x_{3u}^4 = \gamma_2$

d. $\sum_{u=1}^{N} x_{3u}^6 = \gamma_3$

e. $\sum_{u=1}^{N} x_{2u}^2 x_{3u}^2 = \lambda$

Therefore $\mathbf{Z'Z}$ is

$$\mathbf{Z'Z} = \begin{bmatrix}
\theta^2 N & 0 & 0 & 0 & 0 & 0 & \theta\sum_{j=1}^{N} B_{3j} & \theta\sum_{j=1}^{N} B_{4j} & \theta\sum_{j=1}^{N} B_{5j} & 0 \\
 & \sum_{j=1}^{N} A_{1j}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \sum_{j=1}^{N} A_{2j}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & \sum_{j=1}^{N} A_{3j}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \sum_{j=1}^{N} A_{4j}^2 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \sum_{j=1}^{N} A_{5j}^2 & 0 & 0 & 0 & \sum_{j=1}^{N} A_{5j}C_{5j} \\
 & & & & & & \sum_{j=1}^{N} B_{3j}^2 & \sum_{j=1}^{N} B_{3j}B_{4j} & \sum_{j=1}^{N} B_{3j}B_{5j} & 0 \\
 & & & & & & & \sum_{j=1}^{N} B_{4j}^2 & \sum_{j=1}^{N} B_{4j}B_{5j} & 0 \\
 & & & & & & & & \sum_{j=1}^{N} B_{5j}^2 & 0 \\
 & & & & & & & & & \sum_{j=1}^{N} C_{5j}^2
\end{bmatrix}$$

Further, it can be partitioned as

$$\mathbf{Z'Z} = \begin{bmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{L'} & \mathbf{H} \end{bmatrix}$$

The inverse of $\mathbf{Z'Z}$ has been obtained using the following result:

$$\begin{bmatrix} \mathbf{K} & \mathbf{L} \\ \mathbf{L'} & \mathbf{H} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{K}^{-1} + \mathbf{F}\mathbf{E}^{-1}\mathbf{F'} & -\mathbf{F}\mathbf{E}^{-1} \\ -\mathbf{E}^{-1}\mathbf{F'} & \mathbf{E}^{-1} \end{bmatrix}$$

where,

$$\mathbf{E} = \mathbf{H} - \mathbf{L'}\mathbf{K}^{-1}\mathbf{L}$$

$$\mathbf{F} = \mathbf{K}^{-1}\mathbf{L}$$

$$\mathbf{K}^{-1} = \begin{bmatrix}
\frac{1}{\theta^2 N} & 0 & 0 & 0 & 0 & 0 \\
 & \frac{1}{\sum_{j=1}^{N} A_{1j}^2} & 0 & 0 & 0 & 0 \\
 & & \frac{1}{\sum_{j=1}^{N} A_{2j}^2} & 0 & 0 & 0 \\
 & & & \frac{1}{\sum_{j=1}^{N} A_{3j}^2} & 0 & 0 \\
 & & & & \frac{1}{\sum_{j=1}^{N} A_{4j}^2} & 0 \\
 & & & & & \frac{1}{\sum_{j=1}^{N} A_{5j}^2}
\end{bmatrix}$$

$$\mathbf{F} = \mathbf{K}^{-1}\mathbf{L} = \begin{bmatrix}
\frac{\sum_{j=1}^{N} B_{3j}}{\theta N} & \frac{\sum_{j=1}^{N} B_{4j}}{\theta N} & \frac{\sum_{j=1}^{N} B_{5j}}{\theta N} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

$$\mathbf{L'K}^{-1}\mathbf{L} = \begin{bmatrix}
\frac{\left(\sum_{j=1}^{N} B_{3j}\right)^2}{N} & \frac{\sum_{j=1}^{N} B_{3j}\sum_{j=1}^{N} B_{4j}}{N} & \frac{\sum_{j=1}^{N} B_{3j}\sum_{j=1}^{N} B_{5j}}{N} & 0 \\
 & \frac{\left(\sum_{j=1}^{N} B_{4j}\right)^2}{N} & \frac{\sum_{j=1}^{N} B_{4j}\sum_{j=1}^{N} B_{5j}}{N} & 0 \\
 & & \frac{\left(\sum_{j=1}^{N} B_{5j}\right)^2}{N} & 0 \\
 & & & \frac{\left(\sum_{j=1}^{N} A_{5j}C_{5j}\right)^2}{\sum_{j=1}^{N} A_{5j}^2}
\end{bmatrix}$$

$$\mathbf{E}=\mathbf{H}-\mathbf{L}'\mathbf{K}^{-1}\mathbf{L}=\begin{bmatrix} \sum_{j=1}^{N}B_{3j}^2-\frac{\left(\sum_{j=1}^{N}B_{3j}\right)^2}{N} & \sum_{j=1}^{N}B_{3j}B_{4j}-\frac{\sum_{j=1}^{N}B_{3j}\sum_{j=1}^{N}B_{4j}}{N} & \sum_{j=1}^{N}B_{3j}B_{5j}-\frac{\sum_{j=1}^{N}B_{3j}\sum_{j=1}^{N}B_{5j}}{N} & 0 \\ & \sum_{j=1}^{N}B_{4j}^2-\frac{\left(\sum_{j=1}^{N}B_{4j}\right)^2}{N} & \sum_{j=1}^{N}B_{4j}B_{5j}-\frac{\sum_{j=1}^{N}B_{4j}\sum_{j=1}^{N}B_{5j}}{N} & 0 \\ & & \sum_{j=1}^{N}B_{5j}^2-\frac{\left(\sum_{j=1}^{N}B_{5j}\right)^2}{N} & 0 \\ & & & \sum_{j=1}^{N}C_{5j}^2-\frac{\left(\sum_{j=1}^{N}A_{5j}C_{5j}\right)^2}{\sum_{j=1}^{N}A_{5j}^2} \end{bmatrix}$$

$$\mathbf{FE}^{-1}=\begin{bmatrix} \sum_{k=1}^{3}E^{k1}\frac{\sum_{j=1}^{N}B_{(k+2)j}}{\theta N} & \sum_{k=1}^{3}E^{k2}\frac{\sum_{j=1}^{N}B_{(k+2)j}}{\theta N} & \sum_{k=1}^{3}E^{k3}\frac{\sum_{j=1}^{N}B_{(k+2)j}}{\theta N} & \sum_{k=1}^{3}E^{k4}\frac{\sum_{j=1}^{N}B_{(k+2)j}}{\theta N} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\sum_{j=1}^{N}A_{5j}C_{5j}E^{44}}{\sum_{j=1}^{N}A_{5j}^2} \end{bmatrix}=\begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f_{64} \end{bmatrix}$$

Here, E^{pq} is the pth row and qth column element of the $\mathbf{E}^{-1}$ matrix.

$$\mathbf{FE}^{-1}\mathbf{F}'=\begin{bmatrix} \frac{1}{\theta N}\left[\sum_{k=1}^{3}\sum_{j=1}^{N}B_{(k+2)j}f_{1k}\right] & 0 & 0 & 0 & 0 & \frac{f_{14}\sum_{j=1}^{N}A_{5j}C_{5j}}{\sum_{j=1}^{N}A_{5j}^2} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{f_{64}\sum_{j=1}^{N}A_{5j}C_{5j}}{\sum_{j=1}^{N}A_{5j}^2} \end{bmatrix}=\begin{bmatrix} \eta & 0 & 0 & 0 & 0 & \psi \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi \end{bmatrix}$$

$$
(\mathbf{Z}'\mathbf{Z})^{-1} = \begin{bmatrix}
\frac{1}{\theta^2 N}+\eta & 0 & 0 & 0 & 0 & \psi & -f_{11} & -f_{12} & -f_{13} & -f_{14} \\
 & \frac{1}{\sum_{j=1}^{N} A_{1j}^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & \frac{1}{\sum_{j=1}^{N} A_{2j}^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & \frac{1}{\sum_{j=1}^{N} A_{3j}^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & \frac{1}{\sum_{j=1}^{N} A_{4j}^2} & 0 & 0 & 0 & 0 & 0 \\
 & & & & & \frac{1}{\sum_{j=1}^{N} A_{5j}^2}+\xi & 0 & 0 & 0 & -f_{64} \\
 & & & & & & E^{11} & E^{12} & E^{13} & E^{14} \\
 & & & & & & & E^{22} & E^{23} & E^{24} \\
 & & & & & & & & E^{33} & E^{34} \\
 & & & & & & & & & E^{44}
\end{bmatrix}
$$

The variance and covariance of the parameter estimates are

$$V(\hat{\beta}) = \sigma^2 (Z'Z)^{-1}$$

$$V(\hat{\beta}_0) = \left(\frac{1}{\theta^2 N}+\eta\right)\sigma^2$$

$$V(\hat{\beta}_1) = \left(\frac{1}{\sum_{j=1}^{N} A_{1j}^2}\right)\sigma^2$$

$$V(\hat{\beta}_2) = \left(\frac{1}{\sum_{j=1}^{N} A_{2j}^2}\right)\sigma^2$$

$$V(\hat{\beta}_3) = \left(\frac{1}{\sum_{j=1}^{N} A_{3j}^2}\right)\sigma^2$$

$$V(\hat{\beta}_4) = \left(\frac{1}{\sum_{j=1}^{N} A_{4j}^2}\right)\sigma^2$$

$$V(\hat{\beta}_5) = \left(\frac{1}{\sum_{j=1}^{N} A_{5j}^2}+\xi\right)\sigma^2$$

$$V(\hat{\beta}_{33}) = E^{11}\sigma^2$$
$$V(\hat{\beta}_{44}) = E^{22}\sigma^2$$
$$V(\hat{\beta}_{55}) = E^{33}\sigma^2$$
$$V(\hat{\beta}_{555}) = E^{44}\sigma^2$$
$$Cov(\hat{\beta}_0, \hat{\beta}_5) = \psi\sigma^2$$
$$Cov(\hat{\beta}_0, \hat{\beta}_{33}) = -f_{11}\sigma^2$$
$$Cov(\hat{\beta}_0, \hat{\beta}_{44}) = -f_{12}\sigma^2$$
$$Cov(\hat{\beta}_0, \hat{\beta}_{55}) = -f_{13}\sigma^2$$
$$Cov(\hat{\beta}_0, \hat{\beta}_{555}) = -f_{14}\sigma^2$$
$$Cov(\hat{\beta}_5, \hat{\beta}_{555}) = -f_{64}\sigma^2$$
$$Cov(\hat{\beta}_{33}, \hat{\beta}_{44}) = E^{12}\sigma^2$$
$$Cov(\hat{\beta}_{33}, \hat{\beta}_{55}) = E^{13}\sigma^2$$
$$Cov(\hat{\beta}_{33}, \hat{\beta}_{555}) = E^{14}\sigma^2$$
$$Cov(\hat{\beta}_{44}, \hat{\beta}_{55}) = E^{23}\sigma^2$$
$$Cov(\hat{\beta}_{44}, \hat{\beta}_{555}) = E^{24}\sigma^2$$
$$Cov(\hat{\beta}_{55}, \hat{\beta}_{555}) = E^{34}\sigma^2$$

The estimated response at point x_0 is

$$\hat{y}_0 = \hat{\beta}_0 + \sum_{i=1}^{5}\hat{\beta}_i x_0 + \sum_{i=3}^{5}\hat{\beta}_{ii} x_0^2 + \hat{\beta}_{555} x_0^3$$

with the variance of the estimated response as

$$
\begin{aligned}
V(\hat{y}_0) &= \mathbf{x}_0' V(\hat{\boldsymbol{\beta}})\mathbf{x}_0 = \sigma^2 \mathbf{x}_0'(\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{x}_0 \\
&= V(\hat{\beta}_0) + V(\hat{\beta}_1)x_{10}^2 + V(\hat{\beta}_2)x_{20}^2 + V(\hat{\beta}_3)x_{30}^2 + \\
&\quad V(\hat{\beta}_4)x_{40}^2 + V(\hat{\beta}_5)x_{50}^2 + V(\hat{\beta}_{33})x_{30}^4 + V(\hat{\beta}_{44})x_{40}^4 + \\
&\quad V(\hat{\beta}_{55})x_{50}^4 + V(\hat{\beta}_{555})x_{50}^6 + 2Cov(\hat{\beta}_0, \hat{\beta}_{33})x_{30}^2 + \\
&\quad 2Cov(\hat{\beta}_0, \hat{\beta}_{44})x_{40}^2 + 2Cov(\hat{\beta}_0, \hat{\beta}_{55})x_{50}^2 + \\
&\quad 2Cov(\hat{\beta}_5, \hat{\beta}_{555})x_{50}^4 + 2Cov(\hat{\beta}_{33}, \hat{\beta}_{44})x_{30}^2 x_{40}^2 + \\
&\quad 2Cov(\hat{\beta}_{33}, \hat{\beta}_{55})x_{30}^2 x_{50}^2 + 2Cov(\hat{\beta}_{44}, \hat{\beta}_{55})x_{40}^2 x_{50}^2
\end{aligned}
$$

$$
V(\hat{y}_0) = \sigma^2 \left\{
\begin{aligned}
&\left(\frac{1}{\theta^2 N} + \text{ç}\right) + \left(\frac{1}{\sum_{j=1}^{N} A_{1j}^2}\right)x_{10}^2 + \left(\frac{1}{\sum_{j=1}^{N} A_{2j}^2}\right)x_{20}^2 + \\
&\left(\frac{1}{\sum_{j=1}^{N} A_{3j}^2}\right)x_{30}^2 + \left(\frac{1}{\sum_{j=1}^{N} A_{4j}^2}\right)x_{40}^2 + \left(\frac{1}{\sum_{j=1}^{N} A_{5j}^2} + \xi\right)x_{50}^2 + \\
&E^{11}x_{30}^4 + E^{22}x_{40}^4 + E^{33}x_{50}^4 + E^{44}x_{50}^6 - 2f_{11}x_{30}^2 - 2f_{12}x_{40}^2 - \\
&2f_{13}x_{50}^2 - 2f_{64}x_{50}^4 + 2E^{12}x_{30}^2 x_{40}^2 + 2E^{13}x_{30}^2 x_{50}^2 + 2E^{23}x_{40}^2 x_{50}^2
\end{aligned}
\right\}
$$

It is clear that the variance of the predicted response at any point x depends on the distance of that point from the design center.

3. METHODS OF CONSTRUCTION

3.1 Method of Constructing $s_1^{n_1} \times s_2^{n_2} \times s_3^{n_3}$

Consider n_1 (> 1) factors having s_1 ($\geq$ 2) levels, n_2 (> 1) factors having s_2 ($\geq$ 2) levels, and n_3 (> 1) factors having s_3 ($\geq$ 2) levels. The $n_1 + n_2 + n_3$ columns of **X** corresponding to the design given as

$$X_1, X_2, ..., X_{n_1}, X_{n_1+1}, X_{n_1+2}, ..., X_{n_1+n_2}, X_{n_1+n_2+1}, ..., X_{n_1+n_2+n_3}$$

Here, N = max(n_1, n_2, n_3)$\times(s_1^{n_1} \times s_2^{n_2} \times s_3^{n_3})$ points are developed as follows:

$$\mathbf{X} = \begin{bmatrix} \mathbf{O}_{N\times n_1} & \mathbf{Q}_{N\times n_2} & \mathbf{D}_{N\times n_3} \end{bmatrix}$$

where,

$$\mathbf{O} = \begin{bmatrix} o_1 & o_2 & . & . & . & o_{n_1} \\ o_{n_1} & o_1 & . & . & . & o_{n_1-1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ o_2 & o_3 & . & . & . & o_1 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} q_1 & q_2 & . & . & . & q_{n_2} \\ q_{n_2} & q_1 & . & . & . & q_{n_2-1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ q_2 & q_3 & . & . & . & q_1 \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} d_1 & d_2 & . & . & . & d_{n_3} \\ d_{n_3} & d_1 & . & . & . & d_{n_3-1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ d_2 & d_3 & . & . & . & d_1 \end{bmatrix}$$

$$\mathbf{o_i} = \mathbf{1}_{s_1^p \times s_2^{n2} \times s_3^{n3}} \otimes (\text{column vector of } s_1 \text{ levels}) \otimes \mathbf{1}_{s_1^{n1-i}},$$

$p = i - 1, i = 1, 2, \ldots, n_1$

$$\mathbf{q}_j = \mathbf{1}_{s_1^{n1} \times s_2^{q} \times s_3^{n3}} \otimes (\text{column vector of } s_2 \text{ levels}) \otimes \mathbf{1}_{s_2^{n2-j}},$$

$q = j - 1, j = 1, 2, \ldots, n_2$

$$\mathbf{d}_k = \mathbf{1}_{s_1^{n1} \times s_2^{n2} \times s_3^{r}} \otimes (\text{column vector of } s_3 \text{ levels}) \otimes \mathbf{1}_{s_3^{n3-k}},$$

$r = k - 1, k = 1, 2, \ldots, n_3$

If $n_3 = 1$, then $\mathbf{D} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_1 \end{bmatrix}$

The remaining columns of **X** are generated according to the model and values of s_1, s_2, and s_3.

Remark: Here, the number of levels of one factor should not be a multiple of the others.

Example 3.1: Let $n_1 = n_2 = n_3 = 2$ with $s_1 = 2$(1, -1), $s_2 = 3$ (1, 0, -1), and $s_3 = 5$ (2, 1, 0, -1, -2), i.e. $2^2 \times 3^2 \times 5^2$. The first six columns of **X** corresponding to X_1, X_2, X_3 X_4 X_5 and X_6 with 2($2^2 \times 3^2 \times 5^2$) = 900 points is as follows:

$$\mathbf{X} = \begin{bmatrix} \mathbf{O}_{900\times2} & \mathbf{Q}_{900\times2} & \mathbf{D}_{900\times2} \end{bmatrix}$$

The remaining columns of **X** are generated as per the specified model with 10 columns (1 X_1 X_2 X_3 X_4 X_5 X_6 X_3^2 X_4^2 X_5^2 X_6^2 X_5^3 X_6^3 X_5^4 X_6^4).

For, $\alpha = 0.3$,

$$\mathbf{Z'Z}=\begin{bmatrix}
4608 & 0 & 0 & 0 & 0 & 0 & 0 & 3072 & 3072 & 9216 & 9216 & 0 & 0 & 31334.4 & 31334.4\\
 & 1044 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & 1044 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & & 1362 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & & & 1362 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & & & & 6008.4 & 0 & 0 & 0 & 0 & 0 & 18990 & 0 & 0 & 0\\
 & & & & & & 6008.4 & 0 & 0 & 0 & 0 & 0 & 18990 & 0 & 0\\
 & & & & & & & 2502 & 2048 & 6144 & 6144 & 0 & 0 & 20889.6 & 20889.6\\
 & & & & & & & & 2502 & 6144 & 6144 & 0 & 0 & 20889.6 & 20889.6\\
 & & & & & & & & & 27802.8 & 18432 & 0 & 0 & 103757 & 62668.8\\
 & & & & & & & & & & 27802.8 & 0 & 0 & 62668.8 & 103757\\
 & & & & & & & & & & & 66600.72 & 0 & 0 & 0\\
 & & & & & & & & & & & & 66600.72 & 0 & 0\\
 & & & & & & & & & & & & & 396609.8 & 213073.9\\
 & & & & & & & & & & & & & & 396609.8
\end{bmatrix}$$

$$(\mathbf{Z'Z})^{-1}=\begin{bmatrix}
0.0053 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0015 & -0.0015 & -0.0028 & -0.0028 & 0 & 0 & 0.0006 & 0.0006\\
 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & & 0.0007 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & & & 0.0007 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & & & & 0.0017 & 0 & 0 & 0 & 0 & 0 & -0.0005 & 0 & 0 & 0\\
 & & & & & & 0.0017 & 0 & 0 & 0 & 0 & 0 & -0.0005 & 0 & 0\\
 & & & & & & & 0.0022 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & & & & & & & 0.0022 & 0 & 0 & 0 & 0 & 0 & 0\\
 & & & & & & & & & 0.0058 & 0 & 0 & 0 & -0.0013 & 0\\
 & & & & & & & & & & 0.0058 & 0 & 0 & 0 & -0.0013\\
 & & & & & & & & & & & 0.0002 & 0 & 0 & 0\\
 & & & & & & & & & & & & 0.0002 & 0 & 0\\
 & & & & & & & & & & & & & 0.0003 & 0\\
 & & & & & & & & & & & & & & 0.0003
\end{bmatrix}$$

Thus, $V(\hat{\beta}_0)=0.0053\sigma^2$, $V(\hat{\beta}_1)=V(\hat{\beta}_2)=0.0010\sigma^2$, $V(\hat{\beta}_3)=V(\hat{\beta}_4)=0.0007\sigma^2$, $V(\hat{\beta}_5)=V(\hat{\beta}_6)=0.0017\sigma^2$, $V(\hat{\beta}_{33})=V(\hat{\beta}_{44})=0.0022\sigma^2$, $V(\hat{\beta}_{55})=V(\hat{\beta}_{66})\ 0.0058\sigma^2$, $V(\hat{\beta}_{555})=V(\hat{\beta}_{666})=0.0002\sigma^2$ and $V(\hat{\beta}_{5555})=V(\hat{\beta}_{6666})=0.0003\sigma^2$.

Also $Cov(\hat{\beta}_0,\hat{\beta}_{33})=Cov(\hat{\beta}_0,\hat{\beta}_{44})=-0.0015\sigma^2$, $Cov(\hat{\beta}_0,\hat{\beta}_{55})=Cov(\hat{\beta}_0,\hat{\beta}_{66})=-0.0028\sigma^2$ $Cov(\hat{\beta}_0,\hat{\beta}_{5555})=Cov(\hat{\beta}_0,\hat{\beta}_{6666})=0.0006\sigma^2$, $Cov(\hat{\beta}_5,\hat{\beta}_{555})=Cov(\hat{\beta}_6,\hat{\beta}_{666})=-0.0005\sigma^2$ and $Cov(\hat{\beta}_{55},\hat{\beta}_{5555})=Cov(\hat{\beta}_{66},\hat{\beta}_{6666})=-0.0013\sigma^2$.

Further, $V(\hat{y}_0)=0.0072\sigma^2$ for all points in **X**. Hence, the design is rotatable.

3.2 Method of Construction of $s_1^{n_1}\times s_2^{n_2}\times s_3$

Consider n_1 (>1) factors having s_1 levels, n_2 factors having s_2 levels, and another factor having s_3 levels. The n_1 + n_1 + 1 columns of **X** corresponding to the design as given below

$$X_1, X_2,\ldots,X_{n_1},X_{n_1+1},X_{n_1+2},\ldots,X_{n_1+n_2},X_{n_1+n_2+1}$$

Here, N = max(n_1, n_2)$\times(s_1^{n_1}\times s_2^{n_2}\times s_3)$ points are developed as follows:

$$\mathbf{X}=\left[\mathbf{O}_{N\times n_1}\quad \mathbf{Q}_{N\times n_2}\quad \mathbf{D}_{N\times 1}\right]$$

The remaining columns of **X** are generated according to the model and values of s_1, s_2 and s_3.

Remark: Here, s_1 (s_2) should not be a multiple of s_2 (s_1).

Example 3.2: Let n_1 = n_2 = 2 with s_1 = 2 (1, -1), s_2 = 3 (1, 0, -1) and s_3 = 4 (3, 1, -1, -3) i.e. $2^2\times 3^2\times 4$. The first five columns of **X** corresponding to X_1, X_2, X_3 X_4 and X_5 with 2($2^2\times 3^2\times 4$) = 288 points is as follows:

$$\mathbf{X}=\left[\mathbf{O}_{288\times 2}\quad \mathbf{Q}_{288\times 2}\quad \mathbf{D}_{288\times 1}\right]$$

The remaining columns of **X** are generated based on the model with 10 columns (1 X_1 X_2 X_3 X_4 X_5 X_3^2 X_4^2 X_5^2 X_5^3). For, α = 0.5,

$$\mathbf{Z'Z}=\begin{bmatrix}1152&0&0&0&0&0&768&768&5760&0\\0&144&0&0&0&0&0&0&0&0\\0&0&144&0&0&0&0&0&0&0\\0&0&0&264&0&0&0&0&0&0\\0&0&0&0&264&0&0&0&0&0\\0&0&0&0&0&5616&0&0&0&45936\\768&0&0&0&0&0&600&512&3840&0\\768&0&0&0&0&0&512&600&3840&0\\5760&0&0&0&0&0&3840&3840&46848&0\\0&0&0&0&0&45936&0&0&0&407664\end{bmatrix}$$

and

$$\left(\mathbf{Z'Z}\right)^{-1}=\begin{bmatrix}0.01&0&0&0&0&0&-0.007&-0.007&-0.0003&0\\0&0.006&0&0&0&0&0&0&0&0\\0&0&0.006&0&0&0&0&0&0&0\\0&0&0&0.004&0&0&0&0&0&0\\0&0&0&0&0.004&0&0&0&0&0\\0&0&0&0&0&0.002&0&0&0&-0.0003\\-0.007&0&0&0&0&0&0.011&0&0&0\\-0.007&0&0&0&0&0&0&0.011&0&0\\-0.0003&0&0&0&0&0&0&0&0.0001&0\\0&0&0&0&0&-0.0003&0&0&0&0.00003\end{bmatrix}$$

Thus, $V(\hat\beta_0)=0.01\sigma^2$, $V(\hat\beta_1)=V(\hat\beta_2)=0.006\sigma^2$, $V(\hat\beta_3)=V(\hat\beta_4)=0.004\sigma^2$, $V(\hat\beta_5)=0.002\sigma^2$, $V(\hat\beta_{33})=V(\hat\beta_{44})=0.011\sigma^2$, $V(\hat\beta_{55})=0.0001\sigma^2$ and $V(\hat\beta_{555})=0.00003\sigma^2$.

Also, $Cov(\hat\beta_0,\hat\beta_{33})=Cov(\hat\beta_0,\hat\beta_{44})=-0.007\sigma^2$, $Cov(\hat\beta_0,\hat\beta_{55})=-0.0003\ \sigma^2$.

Variance of predicted response, $V(\hat y_0)=0.0275\sigma^2$ for all the points contained in **X**. Hence, the design is rotatable.

Table 1. List of designs and the variances of the estimated response for different values of α

	n_1	n_2	n_3	s_1	s_2	s_3	α					
							0	**0.1**	**0.3**	**0.5**	**0.7**	**0.9**
1	2	2	1	2	3	4	0.0347	0.0319	0.0297	0.0275	0.0224	0.0166
2	2	2	1	2	3	5	0.0306	0.0274	0.0249	0.0227	0.0185	0.0136
3	2	2	1	2	3	6	0.0278	0.0245	0.0217	0.0195	0.0158	0.0116
4	2	2	1	3	4	2	0.0208	0.0190	0.0154	0.0120	0.0091	0.0069
5	2	2	1	3	4	5	0.0104	0.0090	0.0070	0.0053	0.0040	0.0030
6	2	2	1	3	4	6	0.0093	0.0079	0.0061	0.0046	0.0034	0.0026
7	2	2	1	3	5	2	0.0156	0.0140	0.0111	0.0085	0.0064	0.0048
8	2	2	1	3	5	4	0.0089	0.0077	0.0060	0.0045	0.0034	0.0026
9	2	2	1	3	5	6	0.0067	0.0057	0.0043	0.0032	0.0024	0.0018
10	2	2	1	4	5	2	0.0100	0.0089	0.0069	0.0051	0.0038	0.0028
11	2	2	1	4	5	3	0.0071	0.0062	0.0048	0.0035	0.0026	0.0019
12	2	2	1	4	5	6	0.0042	0.0035	0.0026	0.0019	0.0014	0.0010
13	3	2	1	2	3	4	0.0127	0.0115	0.0096	0.0078	0.0061	0.0047
14	3	2	1	2	3	5	0.0111	0.0098	0.0080	0.0065	0.0050	0.0039
15	2	2	2	2	3	5	0.0083	0.0079	0.0072	0.0063	0.0050	0.0037
16	2	2	2	3	4	5	0.0026	0.0024	0.0020	0.0015	0.0011	0.0009

A list of higher-order rotatable mixed-level response surface designs is given in Table 1 for different factor-level combinations.

It is observed that in the presence of the neighbour effect, an increase in the value of α leads to a decrease in the estimated response, which eventually helps in reducing experimental error.

3.3 Comparison with Smaller Design

It has been seen in the previous section that the run size requirement for a rotatable design under the model structure given in equations 2.1 to 2.3 is large. Instead of using the rotatable design if someone goes for a design with a smaller design (Hickernell, 1998), with the same asymmetric factorial case given above in Example 4.1 shown as below, where there are 6 factors denoted by A, B, C, D, E and F with levels as 2, 3, and 5 for subsequent pairs of factors:

Runs	A	B	C	D	E	F	Runs	A	B	C	D	E	F	Runs	A	B	C	D	E	F
1	1	1	-1	1	2	-2	**11**	1	-1	0	1	1	0	**21**	1	1	-1	0	1	2
2	1	1	1	1	-2	2	**12**	-1	-1	1	-1	2	-1	**22**	-1	-1	-1	1	1	1
3	-1	-1	-1	0	-2	2	**13**	1	-1	-1	-1	2	1	**23**	-1	1	0	0	1	-1
4	-1	1	1	-1	0	1	**14**	1	-1	0	1	-1	-1	**24**	1	-1	1	0	0	-1
5	1	1	-1	0	0	0	**15**	1	-1	1	-1	-1	2	**25**	-1	1	0	-1	2	2
6	-1	1	1	0	-1	-2	**16**	1	1	0	1	-1	1	**26**	-1	-1	-1	0	-1	0
7	-1	-1	0	1	0	2	**17**	-1	-1	-1	-1	0	-2	**27**	1	-1	0	0	2	-2
8	-1	1	-1	1	-1	-1	**18**	1	-1	1	0	-2	1	**28**	-1	1	1	1	2	0
9	-1	1	0	0	-2	1	**19**	1	1	-1	-1	-2	-1	**29**	1	1	0	-1	0	-2
10	-1	-1	1	1	1	-2	**20**	1	1	1	-1	1	0	**30**	-1	-1	0	-1	-2	0

In this design, we have incorporated neighbour effects ($\alpha = 0.3$), the $\mathbf{Z'Z}$ and $(\mathbf{Z'Z})^{-1}$ are given below:

$$\mathbf{Z'Z} = \begin{bmatrix}
76.8 & 0 & 0 & 0 & 0 & 0 & 0 & 51.2 & 51.2 & 153.6 & 153.6 & 0 & 0 & 522.24 & 522.24 \\
 & 33.36 & -1.12 & 2.52 & -8.1 & -1.38 & 1.08 & 3.24 & 4.98 & 11.34 & -4.44 & -2.46 & 8.28 & 42.3 & -18.84 \\
 & & 32.64 & -1.56 & 0.18 & -10.68 & 3.6 & 3.24 & -2.94 & -7.2 & 3.72 & -51 & 10.8 & -21.6 & 26.76 \\
 & & & 12.8 & -0.4 & 1.95 & -5.22 & -1.2 & 0.94 & -0.93 & 1.02 & 9.15 & -14.04 & -0.93 & 7.86 \\
 & & & & 21.86 & 0.12 & 0.09 & -1.96 & -0.18 & -4.9 & 14.37 & -2.04 & 0.09 & -33.22 & 63.33 \\
 & & & & & 80.22 & -17.77 & -3.21 & 0.58 & 2.28 & -19.93 & 267.06 & -52.51 & -2.76 & -74.77 \\
 & & & & & & 54.36 & 6.24 & -0.63 & -5.89 & 15.42 & -64.45 & 198.54 & -35.41 & 53.94 \\
 & & & & & & & 39.68 & 32.86 & 104.83 & 102.64 & -19.77 & 20.1 & 360.67 & 347.32 \\
 & & & & & & & & 41.54 & 107.92 & 102.49 & 6.46 & -2.79 & 370.24 & 348.25 \\
 & & & & & & & & & 416.34 & 310.63 & 6.96 & -21.31 & 1537.38 & 1067.71 \\
 & & & & & & & & & & 371.76 & -51.97 & 46.92 & 1066.39 & 1344.12 \\
 & & & & & & & & & & & 1050.06 & -189.91 & 1.92 & -181.69 \\
 & & & & & & & & & & & & 802.8 & -138.67 & 145.92 \\
 & & & & & & & & & & & & & 5837.94 & 3692.59 \\
 & & & & & & & & & & & & & & 5003.52
\end{bmatrix}$$

$$(\mathbf{Z'Z})^{-1} = \begin{bmatrix}
0.7889 & 0.0850 & -0.0470 & -0.0298 & 0.1053 & 0.0304 & 0.0800 & -0.2524 & -0.1418 & -0.4167 & -0.3487 & -0.0130 & -0.0018 & 0.0899 & 0.0597 \\
 & 0.0494 & -0.0029 & -0.0107 & 0.0218 & 0.0120 & 0.0156 & -0.0433 & -0.0317 & -0.0647 & 0.0051 & -0.0027 & -0.0027 & 0.0140 & -0.0016 \\
 & & 0.0431 & 0.0069 & -0.0082 & -0.0064 & 0.0067 & -0.0150 & 0.0060 & 0.0432 & 0.0472 & 0.0036 & -0.0026 & -0.0089 & -0.0100 \\
 & & & 0.0910 & -0.0028 & -0.0057 & 0.0308 & 0.0119 & -0.0063 & 0.0422 & 0.0126 & 0.0016 & -0.0068 & -0.0087 & -0.0036 \\
 & & & & 0.0806 & -0.0045 & 0.0104 & 0.0033 & -0.0071 & -0.0964 & -0.0498 & 0.0009 & 0.0001 & 0.0218 & 0.0061 \\
 & & & & & 0.1052 & 0.0098 & -0.0516 & 0.0020 & -0.0633 & 0.0367 & -0.0263 & -0.0010 & 0.0138 & -0.0056 \\
 & & & & & & 0.2426 & -0.0740 & -0.0294 & -0.0048 & 0.0114 & 0.0007 & -0.0565 & 0.0022 & -0.0058 \\
 & & & & & & & 0.3083 & 0.0645 & 0.0674 & -0.0388 & 0.0143 & 0.0112 & -0.0168 & 0.0090 \\
 & & & & & & & & 0.1766 & -0.0059 & -0.0125 & -0.0008 & 0.0063 & -0.0004 & 0.0030 \\
 & & & & & & & & & 0.5944 & 0.0502 & 0.0148 & -0.0088 & -0.1291 & -0.0053 \\
 & & & & & & & & & & 0.5410 & -0.0017 & -0.0128 & -0.0099 & -0.1076 \\
 & & & & & & & & & & & 0.0079 & -0.0004 & -0.0032 & -0.0001 \\
 & & & & & & & & & & & & 0.0150 & 0.0018 & 0.0031 \\
 & & & & & & & & & & & & & 0.0285 & 0.0008 \\
 & & & & & & & & & & & & & & 0.0224
\end{bmatrix}$$

In this case, the prediction variance is unique for each design point, resulting in 30 distinct types of variances of the estimated response. The corresponding average variances for different values of α are presented in Table 2.

Table 2. Average variances of the estimated response for different α values

α	Average Variance of the estimated response
0	0.5000
0.1	0.5117
0.3	0.6567
0.5	1.0356
0.7	0.7504
0.9	0.4444

From these results, it is evident that although a smaller-run design can be employed, such a design will not be rotatable. This is because, in the current setting, 30 different types of variances of the estimated response are obtained for each α, whereas in a rotatable design, only a single type of variance would be observed. Moreover, the trend in Table 2 indicates that the average variance of the estimated response increases with α up to 0.5, after which it begins to decrease as α continues to increase. This behaviour highlights the influence of the neighbour coefficient (α) on the variability of the estimated response and emphasizes the trade-off between design size and rotatability.

4. R-PACKAGE

To make these designs handier and widen their applicability, R functions viz., asym3() and asym4() have been developed and included in a pre-existing package named "*rsdNE*" (https://cran.r-project.org/package=rsdNE). Snapshots of the R-functions provided in the Appendix.

5. DISCUSSIONS

In this article, a successful attempt has been made to construct rotatable response surface designs that incorporate overlap (neighbour) effects from adjacent left and right-side units for mixed-levels of factors. A general procedure for the construction of the form $s_1^{n_1} \times s_2^{n_2} \times s_3^{n_3}$ is given. Furthermore, in a particular case when $n_3 = 1$, a methodology has been developed. For these designs, it is observed that with an increase in the value of α, the variance of the estimated response decreases accordingly. A comparison with a smaller-run design further demonstrates a complete departure from rotatability. To support the construction and evaluation of these designs, dedicated R functions have been developed to generate the designs and compute their corresponding moment matrices. Looking ahead, exploring higher-order measures of rotatability and extending the framework to saturated or nearly saturated designs with both-sided neighbour effects represent promising directions for future research.

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Conflict of Interest

On behalf of all the authors, the corresponding author declares that there are no conflicts of interest.

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APPENDIX

```
> library(rsdNE)
> asym3(2,2,3,2,4,0.3)
$`X Matrix`
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
 [1,]    1   -1   -1   -1   -1   -3    1    1    9   -27
 [2,]    1    1    1    1    1    3    1    1    9    27
 [3,]    1    1   -1    1    0    3    1    0    9    27
 [4,]    1   -1    1    1   -1    3    1    1    9    27
 [5,]    1   -1   -1    0    1    3    0    1    9    27
 [6,]    1    1    1    0    0    3    0    0    9    27
 [7,]    1    1   -1    0   -1    3    0    1    9    27
 [8,]    1   -1    1   -1    1    3    1    1    9    27
 [9,]    1   -1   -1   -1    0    3    1    0    9    27
[10,]    1    1    1   -1   -1    3    1    1    9    27
[11,]    1    1   -1    1    1    3    1    1    9    27
[12,]    1   -1    1    1    0    3    1    0    9    27
[13,]    1   -1   -1    1   -1    3    1    1    9    27
[14,]    1    1    1    0    1    3    0    1    9    27
[15,]    1    1   -1    0    0    3    0    0    9    27
[16,]    1   -1    1    0   -1    3    0    1    9    27
[17,]    1   -1   -1   -1    1    3    1    1    9    27
[18,]    1    1    1   -1    0    3    1    0    9    27
[19,]    1    1   -1   -1   -1    3    1    1    9    27
[20,]    1   -1    1    1    1    3    1    1    9    27
```

```
$`Inv(Z_prime_Z) matrix`
        [,1]  [,2]  [,3]  [,4]  [,5]  [,6]   [,7]   [,8] [,9] [,10]
 [1,]  0.016 0.000 0.000 0.000 0.000 0.000 -0.009 -0.009    0     0
 [2,]  0.000 0.006 0.000 0.000 0.000 0.000  0.000  0.000    0     0
 [3,]  0.000 0.000 0.006 0.000 0.000 0.000  0.000  0.000    0     0
 [4,]  0.000 0.000 0.000 0.005 0.000 0.000  0.000  0.000    0     0
 [5,]  0.000 0.000 0.000 0.000 0.005 0.000  0.000  0.000    0     0
 [6,]  0.000 0.000 0.000 0.000 0.000 0.004  0.000  0.000    0     0
 [7,] -0.009 0.000 0.000 0.000 0.000 0.000  0.014  0.000    0     0
 [8,] -0.009 0.000 0.000 0.000 0.000 0.000  0.000  0.014    0     0
 [9,]  0.000 0.000 0.000 0.000 0.000 0.000  0.000  0.000    0     0
[10,]  0.000 0.000 0.000 0.000 0.000 0.000  0.000  0.000    0     0

$`Total number of runs`
[1] 288

$`Variance of estimated response`
[1] 0.0297
```

```
> asym4(2,2,3,2,5,2,0.1)
$`X Matrix`
      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12] [,13] [,14] [,15]
 [1,]    1   -1   -1   -1   -1   -2   -2    1    1     4     4    -8    -8    16    16
 [2,]    1    1    1    1    1    2    2    1    1     4     4     8     8    16    16
 [3,]    1    1   -1    1    0    2    1    1    0     4     1     8     1    16     1
 [4,]    1   -1    1    1   -1    2    0    1    1     4     0     8     0    16     0
 [5,]    1   -1   -1    0    1    2   -1    0    1     4     1     8    -1    16     1
 [6,]    1    1    1    0    0    2   -2    0    0     4     4     8    -8    16    16
 [7,]    1    1   -1    0   -1    1    2    0    1     1     4     1     8     1    16
 [8,]    1   -1    1   -1    1    1    1    1    1     1     1     1     1     1     1
 [9,]    1   -1   -1   -1    0    1    0    1    0     1     0     1     0     1     0
[10,]    1    1    1   -1   -1    1   -1    1    1     1     1     1    -1     1     1
[11,]    1    1   -1    1    1    1   -2    1    1     1     4     1    -8     1    16
[12,]    1   -1    1    1    0    0    2    1    0     0     4     0     8     0    16
[13,]    1   -1   -1    1   -1    0    1    1    1     0     1     0     1     0     1
[14,]    1    1    1    0    1    0    0    0    1     0     0     0     0     0     0
[15,]    1    1   -1    0    0    0   -1    0    0     0     1     0    -1     0     1
[16,]    1   -1    1    0   -1    0   -2    0    1     0     4     0    -8     0    16
[17,]    1   -1   -1   -1    1   -1    2    1    1     1     4    -1     8     1    16
[18,]    1    1    1   -1    0   -1    1    1    0     1     1    -1     1     1     1
[19,]    1    1   -1   -1   -1   -1    0    1    1     1     0    -1     0     1     0
[20,]    1   -1    1    1    1   -1   -1    1    1     1     1    -1    -1     1     1
[21,]    1   -1   -1    1    0   -1   -2    1    0     1     4    -1    -8     1    16
```

```
$`Inv(Z_prime_Z) matrix`
        [,1]  [,2]  [,3]  [,4]  [,5]   [,6]   [,7]   [,8]   [,9]  [,10]  [,11]  [,12]  [,13]  [,14]  [,15]
 [1,]  0.007 0.000 0.000 0.000 0.000  0.000  0.000 -0.002 -0.002 -0.003 -0.003  0.000  0.000  0.001  0.001
 [2,]  0.000 0.001 0.000 0.000 0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000
 [3,]  0.000 0.000 0.001 0.000 0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000
 [4,]  0.000 0.000 0.000 0.001 0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000
 [5,]  0.000 0.000 0.000 0.000 0.001  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000
 [6,]  0.000 0.000 0.000 0.000 0.000  0.002  0.000  0.000  0.000  0.000  0.000 -0.001  0.000  0.000  0.000
 [7,]  0.000 0.000 0.000 0.000 0.000  0.000  0.002  0.000  0.000  0.000  0.000  0.000 -0.001  0.000  0.000
 [8,] -0.002 0.000 0.000 0.000 0.000  0.000  0.000  0.002  0.000  0.000  0.000  0.000  0.000  0.000  0.000
 [9,] -0.002 0.000 0.000 0.000 0.000  0.000  0.000  0.000  0.002  0.000  0.000  0.000  0.000  0.000  0.000
[10,] -0.003 0.000 0.000 0.000 0.000  0.000  0.000  0.000  0.000  0.007  0.000  0.000  0.000 -0.001  0.000
[11,] -0.003 0.000 0.000 0.000 0.000  0.000  0.000  0.000  0.000  0.000  0.007  0.000  0.000  0.000 -0.001
[12,]  0.000 0.000 0.000 0.000 0.000 -0.001  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000
[13,]  0.000 0.000 0.000 0.000 0.000  0.000 -0.001  0.000  0.000  0.000  0.000  0.000  0.000  0.000  0.000
[14,]  0.001 0.000 0.000 0.000 0.000  0.000  0.000  0.000  0.000 -0.001  0.000  0.000  0.000  0.000  0.000
[15,]  0.001 0.000 0.000 0.000 0.000  0.000  0.000  0.000  0.000  0.000 -0.001  0.000  0.000  0.000  0.000

$`Total number of runs`
[1] 1800

$`Variance of estimated response`
[1] 0.0079
```