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Bayesian A-Optimal Designs for Gamma and Poisson Regression Models: An Algorithmic Approach

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SUMMARY

Bayesian design approaches leverage prior information about unknown parameters to improve the efficiency of experimental designs. In this study, we propose two algorithms for deriving Bayesian A-optimal designs for Gamma and Poisson regression models involving two explanatory factors. The experimental region is assumed to be a unit rectangle. The present work focuses specifically on vertex-type designs, where the support points of the initial designs are restricted to the vertices of the experimental region. Initially, the Fisher information matrix is computed based on the support points of the proposed locally optimal designs. Subsequently, Bayesian A-optimal designs are determined by incorporating prior distributions-namely, Uniform, Normal, Beta, and Gamma-on the unknown parameters. The corresponding optimal weights for each design are computed using MATLAB.

Keywords: Fisher information matrix; Bayesian A-optimal design; Gamma regression model; Poisson regression model; Prior information.

AMS Subject Classifications: 62K05

0. Dr. Manindra Nath Das: A Tribute to a Visionary Statistician

Dr. Manindra Nath Das, a renowned statistician, was born on February 1, 1923, in Khalishpur, District Khulna, Bangladesh. He passed away on January 9, 2012, leaving behind a profound legacy in the field of statistics. His birth centenary was commemorated in 2023, and to honor his memory, the Society of Statistics, Computer and Applications (SSCA) organized a special online session, inviting Dr. Das's students and their students to participate. This event coincided with the silver jubilee of SSCA's founding, adding further significance to the occasion. SSCA also published a special issue of its journal Statistics and Applications (S&A) in 2013 [Volume 11, Issues 1 & 2 (New Series)], as a tribute to a legendary statistician.

Dr. Das made pioneering contributions to several areas of statistics, particularly in the design and analysis of experiments. His work displayed remarkable depth and versatility, often addressing challenging problems with innovative solutions. One of his early achievements involved reinforced incomplete block designs, which later proved crucial for control-test comparison experiments. In 1960, he introduced a new class of designs known as circular designs, closely related to cyclic designs. Another of his major contributions was the development of a unified method for constructing confounded designs for asymmetrical factorial experiments. This influential work was followed by numerous widely cited papers in the area. He also developed methods for constructing designs to fit response surfaces, demonstrating how second- and third- order rotatable designs could be obtained using balanced incomplete block designs. His concept of modified rotatable designs found several applications in agricultural experiments.

Dr. Das also made important contributions to the construction of incomplete block designs for biological assays-now considered a classic in the field-and to the design of experiments involving mixtures. His deep understanding of statistical design was complemented by his strong intuitive insights, which often guided him to breakthroughs long before formal algebraic proofs were established.

He co-authored several well-regarded books, including the widely used Design and Analysis of Experiments, written with N.C. Giri. This text remains a standard reference in Indian universities.

In addition to his academic contributions, Dr. Das was the founder of the Society of Statistics, Computer and Applications (SSCA). He also initiated the journal Statistics and Applications and served as its Chair Editor. Through his unwavering dedication, he nurtured both the SSCA and its journal, helping them gain international recognition and respect.

Dr. M.N. Das's intellectual brilliance, deep intuition, and lifelong passion for statistics and computer applications continue to inspire generations of statisticians. His legacy endures in the many students he mentored, the institutions he helped shape, and the enduring relevance of his work.

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1. INTRODUCTION

The generalized linear model (GLM) extends the ordinary linear regression framework to accommodate both continuous and discrete response variables. Introduced by Nelder and Wedderburn (1972), the GLM allows observations from one-parameter exponential family distributions to be related to a set of explanatory variables through appropriate link functions. Models such as Poisson, Gamma, and Logistic regression models represent specific cases within the framework of generalized linear models (GLMs). Due to their flexibility and broad applicability, GLMs have proven to be highly valuable across a wide range of disciplines, including social sciences, agriculture, educational research, insurance, industrial applications, pharmaceutical sciences, and chemical sciences. For further details and applications, readers are referred to the works of Walker and Duncan (1967), Myers and Montgomery (1997), and Goldburd *et al.* (2016), among others.

The fundamental goal of optimal design theory is to develop experimental designs that maximize information efficiency based on a chosen optimality criterion, thereby ensuring that the predicted response closely approximates the true mean response over a specified region of interest. The strategic selection of design points enables experimenters to utilize available resources more effectively, reduce overall costs, and adhere to time constraints.

A central concept in optimal design theory is the Fisher information matrix, which serves as the foundation for deriving optimal designs. In the context of GLMs, the Fisher information matrix is a function of the model parameters. Consequently, optimal designs cannot be determined without some prior knowledge of these parameters. One commonly adopted strategy to address this challenge is the local optimality approach, originally proposed by Chernoff (1953). This method involves deriving an optimal design based on a specified, fixed value of the model parameters-typically regarded as the experimenter's best guess.

After the pioneering work of Chernoff (1953), several works have been published in the domain of non-Bayesian experimental design for GLMs. The basic concept of deriving such designs is to obtain the locally optimum designs based on the best prediction of the value of the associated parameters. Finney (1978), Abdelbasit and Plackett (1983), and Meeker and Hahn (1977) obtained non-Bayesian designs for Logistic regression models. Gaffke *et al*. (2019) provided analytical solutions to derive locally D- and A-optimal designs for the Gamma regression models that involve intercept terms. They also established that the derived designs are essentially a complete class of designs. Idais and Schwabe (2021) found locally D- and A-optimal designs for the Gamma regression models having linear predictors without intercept. Idais (2021) obtained D-, A-, and Kiefer's Φ_k-criteria optimality for vertex-type designs in GLMs. Recently, Panda and Biswal (2025a), Panda *et al.* (2025) obtained locally R-optimal designs for both Logistic and Gamma regression models. In another recent study, Panda and Biswal (2025b) developed locally A-optimal designs for a two-variable Logistic regression model with positive parameters, considering a restricted design space.

In many real-world problems, prior information is often available in the form of historical data, expert opinions, and other relevant sources. The fundamental concept behind the construction of Bayesian optimal designs is the use of a prior distribution on the unknown parameters. By adopting a Bayesian decision framework, all available information can be effectively utilized to develop an improved design. Chaloner and Verdinelli (1995) provided a comprehensive review of Bayesian experimental design. More general experimental designs, including Bayesian optimal designs for GLMs, were discussed by Khuri *et al.* (2006). For further insights into recent advancements in Bayesian optimal designs, readers may refer to the works of Holling and Schwabe (2011) and Ryan *et al.* (2015). A significant challenge in constructing Bayesian optimal designs is the development of efficient computational algorithms. Although several studies [*e.g.*, Abebe *et al.* (2014), Chaloner and Larntz (1989)] have addressed this issue, it remains insufficiently explored in the literature.

The motivation of this study is to develop Bayesian A-optimal designs for Gamma and Poisson regression models, each involving three unknown parameters, using two distinct algorithms. To this end, we consider the support points of the local A-optimal designs for these models as presented by Idais and Schwabe (2021) (*pg* 1890–1894). Within the context of the Bayesian decision framework, we assume a range of priors for the unknown parameters: Uniform, Gamma, Beta, and Normal priors for the Gammar egression model, and a Gamma prior for the Poisson regression model. Furthermore, we demonstrate that the proposed Bayesian A-optimal designs offer superior efficiency when compared to the locally A-optimal designs previously discussed by Idais (2021).

The remainder of the article is organized is as follows. Section 2 provides the GLMs and A-optimal design. In Section 3, we obtain Bayesian A-optimal designs for the Gamma regression model. Section 4 provides the Bayesian A-optimal designs for the Poisson regression model. Design efficiency is discussed in Section 5. Finally, the article is concluded with some conclusions in Section 6.

2. GENERALIZED LINEAR MODELS AND A-OPTIMAL DESIGN

In this section, we first introduce the framework of GLMs, followed by a definition of the A-optimal design, which forms the basis for the main theoretical results.

Let Y denote a univariate response variable. We assume that the distribution of Y belongs to the one-parameter exponential family, characterized by the probability density (or mass) function of the form

$$p(y;\phi) = \exp(y\phi - m(\phi) + n(y)) \tag{1}$$

where $m(.)$, $n(.)$ are the known functions and ϕ is the canonical parameter. In the context of GLMs, each response variable Y is observed for a statistical unit at a given value of a covariate $\boldsymbol{x} = (x_1, x_2, \ldots, x_q)'$ vector where $\boldsymbol{x}$ lies within a specified region $\Omega \subseteq R^q, q \geq 1$. The canonical parameter $\phi = \phi(\boldsymbol{x}, \boldsymbol{\beta})$ varies with the covariate values $\boldsymbol{x}$, while the parameter vector $\boldsymbol{\beta} \in R^q$ remains fixed. The expected value (mean) of the response Y is given by $E(Y) = \mu(\boldsymbol{x}, \boldsymbol{\beta}) = m'(\phi)$ and the variance function $V(\mu(\boldsymbol{x}, \boldsymbol{\beta})) = m''(\phi)$. For a more comprehensive discussion, refer to the work of McCullagh and Nelder (1989).

Let $\boldsymbol{g}(\boldsymbol{x}) : \Omega \to R^q$ denote a vector of continuous and linearly independent functions.The linear predictor is defined as $\eta = \boldsymbol{g}'(\boldsymbol{x})\boldsymbol{\beta}$, where $\boldsymbol{\beta}$ is the parameter vector. In GLM framework, the predictor is related to mean response *via* a differentiable, one-to-one link function f, such that $\eta = f(\mu(\boldsymbol{x}, \boldsymbol{\beta}))$. The intensity or weight function at a point $\boldsymbol{x} \in \Omega$ is expressed as

$$\lambda(\boldsymbol{x}, \boldsymbol{\beta}) = V^{-1}(\mu(\boldsymbol{x}, \boldsymbol{\beta}))\left(\frac{\mathrm{d}\mu(\boldsymbol{x}, \boldsymbol{\beta})}{\mathrm{d}\eta}\right)^2 \tag{2}$$

where $\frac{\mathrm{d}\mu(\boldsymbol{x}, \boldsymbol{\beta})}{\mathrm{d}\eta} = \frac{1}{f'(\mu(\boldsymbol{x}, \boldsymbol{\beta}))}$. Since the link function f is differentiable and monotonic, it follows that $\lambda(\boldsymbol{x}, \boldsymbol{\beta}) > 0$, for all $\boldsymbol{x} \in \Omega$. This function captures the marginal contribution of each design point to the overall Fisher information matrix. The Fisher information matrix at each design point $\boldsymbol{x} \in \Omega$ is denoted by

$$\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{\beta}) = \lambda(\boldsymbol{x}, \boldsymbol{\beta})\boldsymbol{g}(\boldsymbol{x})\boldsymbol{g}'(\boldsymbol{x}). \tag{3}$$

Additional information is available in Atkinson and Woods (2015) and Fedorov and Leonov (2013). Let us consider an approximate (continuous) design defined by

$$\xi = \begin{pmatrix} \boldsymbol{x}_1 & \ldots & \boldsymbol{x}_s \\ w_1 & \ldots & w_s \end{pmatrix}$$

that consists of 's' number of distinct support points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_s \in \Omega$ and associated positive weights $w_1, w_2, \ldots, w_s > 0$, $i = 1, 2, \ldots, s$ such that $\sum_{i=1}^{s} w_i = 1$ [*see* Silvey (1980)]. The Fisher information matrix corresponding to the continuous design ξ, as defined in Equation (3), is denoted by:

$$\boldsymbol{M}(\xi, \boldsymbol{\beta}) = \int_{\Omega} \boldsymbol{M}(\boldsymbol{x}, \boldsymbol{\beta})\xi(\mathrm{d}\boldsymbol{x}) = \sum_{i=1}^{s} w_i \boldsymbol{M}(\boldsymbol{x}_i, \boldsymbol{\beta}) \tag{4}$$

The definition of A-optimal design is presented below.

Definition 1: Let Ξ denotes the set of continuous designs. A design $\xi^* \in \Xi$ with an information matrix $\boldsymbol{M}(\xi^*)$ of the form given in Equation (4) is called A-optimal design if

$$\min_{\xi \in \Xi} \mathrm{Tr}\left(\boldsymbol{M}^{-1}(\xi)\right) = Tr\left(\boldsymbol{M}^{-1}(\xi^*)\right).$$

Here $\mathrm{Tr}\left(\boldsymbol{M}^{-1}(\xi)\right)$ denote the trace of the inverse of the Fisher information matrix.

We consider both Gamma and Poisson regression models involving two factors with linear predictor in the following form:

$$\eta(\boldsymbol{x},\boldsymbol{\beta}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \tag{5}$$

where $\beta_0, \beta_1, \beta_2$ are the model parameters corresponding to the intercept and the two factors x_1 and x_2, respectively.

Definition 2: A design ξ^* for the above-mentioned two-factor model is defined as Bayesian A-optimal if it minimizes the expected trace of the inverse of the information matrix with respect to the prior distribution *i.e.*

$$\xi^* = \underset{\xi}{\mathbf{arg\,min}} \int_{\beta_0}\int_{\beta_1}\int_{\beta_2} \mathbf{Tr}\left(\boldsymbol{M}^{-1}(\xi)\right)\pi(\beta_0)\pi(\beta_1)\pi(\beta_2)\, d\beta_0 d\beta_1 d\beta_2,$$

for all $\xi \in \Xi$. (6)

Here $\pi(\beta_0)\pi(\beta_1)\pi(\beta_2)$, denote the prior distributions of $\beta_0, \beta_1, \beta_2$ respectively.

In the next section, we discuss the computation of Bayesian A-optimal designs for a Gamma regression model involving two factors.

3. BAYESIAN A-OPTIMAL DESIGNS FOR GAMMA REGRESSION MODEL

The Gamma regression model is characterized by $\boldsymbol{g}'(\boldsymbol{x})\boldsymbol{\beta} = \kappa / \lambda(\boldsymbol{x},\boldsymbol{\beta})$ with intensity function $\lambda(\boldsymbol{x},\boldsymbol{\beta}) = \left(\boldsymbol{g}'(\boldsymbol{x})\boldsymbol{\beta}\right)^{-2}$. In this context, it is assumed that the shape parameter $\kappa > 0$ and $\boldsymbol{g}'(\boldsymbol{x})\boldsymbol{\beta} > 0$ for all $\boldsymbol{x} \in \Omega$. We assume that the experimental region be $\Omega = [0,1]^2$. In this case, the optimal designs are supported solely at the vertices of the square design region, specifically at points $\boldsymbol{x}_1^* = (0,0)'$, $\boldsymbol{x}_2^* = (1,0)'$, $\boldsymbol{x}_3^* = (0,1)'$, $\boldsymbol{x}_4^* = (1,1)'$. The corresponding intensity functions at these points are denoted by $\lambda_1 = \boldsymbol{\beta}_0^{-2}$, $\lambda_2 = (\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1)^{-2}$, $\lambda_3 = (\boldsymbol{\beta}_0 + \boldsymbol{\beta}_2)^{-2}$, $\lambda_4 = (\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1 + \boldsymbol{\beta}_2)^{-2}$ respectively [*see* Idais (2021)].

We propose an algorithmic framework *i.e.*, Algorithm 1 for the computation of Bayesian A-optimal designs under this prior specification. The analysis considers a range of prior distributions for the parameters, including Uniform, Gamma, Beta, and Normal distributions, among others. These distributions are denoted by $U(.,.)$, $G(.,.)$, $B(.,.)$, and $N(.,.)$ respectively.

Algorithm 1

Step 1. Input the guess values of $\beta_0, \beta_1, \beta_2$.

Step2. Find the values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ where $\lambda_1 = \dfrac{1}{\beta_0^2}$, $\lambda_2 = \dfrac{1}{(\beta_0+\beta_1)^2}$, $\lambda_3 = \dfrac{1}{(\beta_0+\beta_2)^2}$, $\lambda_4 = \dfrac{1}{(\beta_0+\beta_1+\beta_2)^2}$.

Step3. Define the two-dimensional point $\boldsymbol{x}_1^* = (0,0)'$, $\boldsymbol{x}_2^* = (1,0)'$, $\boldsymbol{x}_3^* = (0,1)'$, $\boldsymbol{x}_4^* = (1,1)'$.

Step 4. Choose three points from the points specified in step 3 by considering the following

thumb rule [Idais (2021), *pg*. 1890] and name them $\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_3$:

(a) Choose $\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \boldsymbol{x}_3^*$ **if** $\beta_0^2 - \beta_1\beta_2 \le 0$

(b) Choose $\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \boldsymbol{x}_4^*$ **if** $(\beta_0+\beta_1)^2 + \beta_1\beta_2 \le 0$

(c) Choose $\boldsymbol{x}_1^*, \boldsymbol{x}_3^*, \boldsymbol{x}_4^*$ **if** $(\beta_0+\beta_2)^2 + \beta_1\beta_2 \le 0$

(d) Choose $\boldsymbol{x}_2^*, \boldsymbol{x}_3^*, \boldsymbol{x}_4^*$ **if** $\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_1\beta_2 + 2\beta_0(\beta_1+\beta_2) \le 0$

(e) Otherwise choose $\boldsymbol{x}_1^*, \boldsymbol{x}_2^*, \boldsymbol{x}_3^*, \boldsymbol{x}_4^*$ **(name them z_1, z_2, z_3, and z_4).**

Step 5. Find trace of the matrix $\boldsymbol{M}^{-1}(\xi)$ i.e. $\mathrm{Tr}(\boldsymbol{M}^{-1}(\xi))$ for the chosen points $\mathbf{z_i}$, $\mathrm{i} = 1, 2, 3, 4$ as the case from step 4 (ref. Appendix II) at the guess values of β_0, β_1, β_2.

Step 6. Evaluate the function corresponding to the multiple integral

$$\psi(\xi^*) = \int_{\beta_0}\int_{\beta_1}\int_{\beta_2} \mathrm{Tr}\left(\boldsymbol{M}^{-1}(\xi)\right)\pi(\beta_0)\pi(\beta_1)\pi(\beta_2)\, d\beta_0 d\beta_1 d\beta_2.$$

when all three prior distributions are of the same type (*i.e.* Uniform, Normal, Gamma, and Beta) distributions.

Step 7. Obtain the numerical values of optimal weights w_1^*, w_2^*, and w_3^* by minimizing the function

$\psi\left(\xi^*\right)$ *w.r.t* w_1, w_2, w_3. For case (e) in Step 4: compute w_4^* as $w_4^* = 1 - (w_1^* + w_2^* + w_3^*)$.

The next section discusses the computation of Bayesian A-optimal designs for a Poisson regression model that includes two factors.

4. BAYESIAN A-OPTIMAL DESIGNS FOR POISSON REGRESSION MODEL

The Poisson regression model is characterized by $g'(x)\beta = \log(\lambda(x,\beta))$ with intensity function $\lambda(x,\beta) = e^{(g'(x)\beta)}$. In this context, it is assumed that $g'(x)\beta > 0$ for all $x \in \Omega$. Here also we assume that the experimental region be $\Omega = [0,1]^2$. In this case, the optimal designs are supported solely at the vertices of the square design region, specifically at points $x_1^* = (0,0)'$, $x_2^* = (1,0)'$, $x_3^* = (0,1)'$, $x_4^* = (1,1)'$. The corresponding intensity functions at these points are denoted by $\lambda_1 = e^{\beta_0}$, $\lambda_2 = e^{(\beta_0+\beta_1)}$, $\lambda_3 = e^{(\beta_0+\beta_2)}$, $\lambda_4 = e^{(\beta_0+\beta_1+\beta_2)}$ respectively [*ref.* Idais (2021)].

To derive Bayesian A-optimal designs for the Poisson regression model, we present Algorithm 2. In this framework, Gamma prior distributions are assumed for all unknown parameters. Algorithm 2 is developed based on Algorithm 1, with the following modifications.

Algorithm 2

Replace Steps 2, 4, and 6 in Algorithm 1 by the following steps:

Step 2. Find the values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ where $\lambda_1 = e^{\beta_0}$, $\lambda_2 = e^{(\beta_0+\beta_1)}$, $\lambda_3 = e^{(\beta_0+\beta_2)}$, $\lambda_4 = e^{(\beta_0+\beta_1+\beta_2)}$.

Step 4. Choose three points from the points specified in step 3 by following thumb rule (*ref. pg.* 1894, Idais (2021)] and name them z_1, z_2, z_3:

(a) Choose x_1^*, x_2^*, x_3^* **if**

$$e^{-\beta_1} + e^{-\beta_2} + e^{-(\beta_1+\beta_2)} + 2\sqrt{\frac{2}{3}}e^{-\frac{(2\beta_1+\beta_2)}{2}} + e^{-\frac{(2\beta_2+\beta_1)}{2}} - 1 \le 0$$

(b) Choose x_1^*, x_2^*, x_4^* **if**

$$e^{\beta_1} + e^{-\beta_2} + e^{(\beta_1-\beta_2)} + e^{(\beta_1-\beta_2/2)} + \sqrt{2}e^{(\beta_1/2-\beta_2)} - 1 \le 0$$

(c) Choose x_1^*, x_3^*, x_4^* **if**

$$e^{\beta_2} + e^{-\beta_1} + e^{(\beta_2-\beta_1)} + e^{(\beta_2-\beta_1/2)} + \sqrt{2}e^{(\beta_2/2-\beta_1)} - 1 \le 0$$

(d) Choose x_2^*, x_3^*, x_4^* **if**

$$e^{\beta_1} + e^{\beta_2} + e^{(\beta_1+\beta_2)} + 2\sqrt{\frac{2}{3}}e^{(\beta_2+\beta_1/2)} + e^{(\beta_1+\beta_2/2)} - 1 \le 0$$

(e) Otherwise choose $x_1^*, x_2^*, x_3^*, x_4^*$ (name them z_1, z_2, z_3, and z_4**).**

Step 6. Obtain the function

$$\psi\left(\xi^*\right) = \int_{\beta_0}\int_{\beta_1}\int_{\beta_2} Tr\left(M^{-1}(\xi)\right)\pi(\beta_0)\pi(\beta_1)\pi(\beta_2)\,d\beta_0 d\beta_1 d\beta_2$$

when all three prior distributions are of the same type (*i.e.* Gamma) distributions.

Remark: In this study, the Gamma prior was selected for constructing Bayesian A-optimal designs within the Poisson regression framework, primarily due to its conjugacy and computational efficiency. This choice enables analytical simplification, particularly in the computation of the expected Fisher information matrix. However, the use of a Gamma prior does not constitute a fundamental restriction. Bayesian A-optimal designs can likewise be developed under alternative, potentially non-conjugate priors such as the log-normal or normal distributions. While these alternatives may offer greater modeling flexibility, they typically involve increased computational demands, making their integration a compelling direction for future research.

5. DESIGN EFFICIENCY

To evaluate the performance of the Bayesian A-optimal design relative to the locally A-optimal design for both regression models, we compute the A-efficiency using the following formula:

$$\text{A}-\text{efficiency} = \left(\frac{\text{Tr}\left(M^{-1}(\xi_A)\right)}{\text{Tr}\left(M^{-1}\left(\xi_A^{Bayes}\right)\right)}\right)^{\frac{1}{p}}.$$

Here, $\text{Tr}\left(M^{-1}(\xi_A)\right)$ and $\text{Tr}\left(M^{-1}\left(\xi_A^{Bayes}\right)\right)$ the trace of the locally A-optimal design (ξ_A) and Bayesian A-optimal design $\left(\xi_A^{Bayes}\right)$ respectively. The parameter p corresponds to the number of unknown parameters in the regression model. The computed A-efficiency values are presented in Tables 21-24 in Appendix I. An A-efficiency value close to 1 indicates that the design performs well with respect to the A-optimality criterion. The results in the aforementioned tables demonstrate that the Bayesian A-optimal designs consistently out perform the locally A-optimal designs in terms of efficiency.

6. CONCLUSION

In this study, Bayesian A-optimal designs are developed for three-parameter Gamma and Poisson regression models employing appropriate link functions. The designs are constructed by formulating and solving a nonlinear optimization problem under various prior distributions on the model parameters. The support points of the corresponding locally A-optimal designs are utilized as a basis for the Bayesian design construction. A comparative analysis demonstrates that Bayesian A-optimal designs consistently outperform their locally A-optimal counterparts, particularly with respect to overall efficiency. These findings underscore the practical benefits of incorporating prior information via Bayesian design criteria in generalized linear models, thereby reinforcing their applicability in scenarios where prior knowledge of model parameters is available.

The focus on Bayesian A-optimal designs for Gamma and Poisson regression models with two explanatory variables is driven by computational tractability and interpretability. As model dimensionality increases, integrating over priors and optimizing complex criteria becomes significantly more challenging. Restricting the design space enables clearer insight and avoids the computational burden of high-dimensional problems. While extension to higher dimensions is theoretically possible, it entails substantial computational effort and remains a topic for future research.

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CONFLICT OF INTEREST

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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APPENDIX-I

Table 1. Bayesian A-optimal design for Gamma regression model [Support points (0, 0), (1, 0), (0, 1) and $\beta_0 \sim U(a,b), \beta_1 \sim U(c,d), \beta_2 \sim U(e,f)$]

a	b	c	d	e	f	w_1	w_2	w_3
0	1	0	1	0	1	0.3164	0.3417	0.3417
0	10	0	1	0	1	0.4457	0.2771	0.2771
0	30	0	1	0	1	0.4579	0.2710	0.2710
0	50	0	1	0	1	0.4603	0.2698	0.2698
0	100	0	1	0	1	0.4622	0.2688	0.2688
0	500	0	1	0	1	0.4637	0.2681	0.2681
0	1	0	10	0	1	0.1205	0.7493	0.1301
0	1	0	20	0	1	0.0710	0.8521	0.0767
0	1	0	50	0	1	0.0318	0.9337	0.0344
0	1	0	100	0	1	0.0165	0.9654	0.0179
0	1	0	500	0	1	0.0034	0.9928	0.0037
0	1	0	1	0	10	0.1205	0.1301	0.7493
0	1	0	1	0	30	0.0504	0.0544	0.8951
0	1	0	1	0	50	0.0318	0.0344	0.9337
0	1	0	1	0	100	0.0165	0.0179	0.9654
0	1	0	1	0	500	0.0034	0.0037	0.9928
20	30	40	50	60	70	0.2138	0.3344	0.4420
125	200	250	300	400	450	0.2167	0.3344	0.4488
500	600	550	675	670	870	0.2773	0.3383	0.3842
690	800	700	900	500	650	0.3105	0.3718	0.3716

Table 2. Bayesian A-optimal design for Gamma regression model, [Support points (0, 0), (1, 0), (0, 1) and $\beta_0 \sim N(\mu_1,\sigma_1), \beta_1 \sim N(\mu_2,\sigma_2), \beta_2 \sim N(\mu_3,\sigma_3)$]

μ_1	σ_1	μ_2	σ_2	μ_3	σ_3	w_1	w_2	w_3
1	15	10	15	10	25	0.3210	0.2946	0.3842
3	10	5	15	10	30	0.2511	0.2739	0.4748
2	5	3	10	5	15	0.2399	0.3151	0.4448
2	20	5	5	10	5	0.4327	0.2706	0.2965
2	30	10	5	5	10	0.4292	0.2293	0.2770
5	2	5	10	20	30	0.1487	0.2277	0.6235
20	25	15	5	15	20	0.3793	0.2962	0.3244
30	2	20	30	15	30	0.3164	0.3545	0.3289
10	20	5	5	5	5	0.4316	0.2841	0.2841
10	30	10	10	5	5	0.4343	0.2976	0.2689
30	10	10	20	10	20	0.3740	0.3129	0.3129
40	50	20	25	50	60	0.3553	0.2627	0.3818
50	80	16	30	40	10	0.4166	0.2752	0.3080
500	805	169	303	407	105	0.4157	0.2759	0.3082
5	8	10	15	4	10	0.2989	0.4147	0.2863
5	1	1	4	4	1	0.3501	0.2886	0.3612
-13	20	-26	15	- 10	30	0.3168	0.3552	0.3279
2	5	3	6	4	5	0.3346	0.3326	0.3326
1	5	2	10	3	15	0.2405	0.3152	0.4441
-5	2	-10	2	-15	6	0.2046	0.3349	0.4603

Table 3. Bayesian A-optimal design for Gamma regression model [Support points (0, 0), (1, 0), (0, 1) and $\beta_0 \sim B(a_1,b_1), \beta_1 \sim B(a_2,b_2), \beta_2 \sim B(a_3,b_3)$]

a_1	b_1	a_2	b_2	a_3	b_3	w_1	w_2	w_3
2	2	2	2	2	2	0.3114	0.3442	0.3442
2	5	2	2	2	2	0.2540	0.3729	0.3729
2	10	2	2	2	2	0.1928	0.4035	0.4035
2	50	2	2	2	2	0.0647	0.4672	0.0.4672
5	2	2	2	2	2	0.3373	0.3313	0.3313
10	2	2	2	2	2	0.3491	0.3254	0.3254
50	2	2	2	2	2	0.3603	0.3198	0.3198
2	2	10	2	2	2	0.2829	0.4043	0.3127
2	2	20	2	2	2	0.2769	0.4168	0.3061
2	2	30	2	2	2	0.2747	0.4215	0.3037
2	2	50	2	2	2	0.2729	0.4253	0.3017
2	2	2	10	2	2	0.3502	0. 2624	0.3872
2	2	2	20	2	2	0.3604	0.2411	0.3984
2	2	2	30	2	2	0.3642	0.2329	0.4027
2	2	2	50	2	2	0.3675	0.2261	0.4063
2	2	2	2	2	5	0.3352	0.3706	0.2941
2	2	2	2	2	10	0.3502	0.3872	0.2624
2	2	2	2	10	2	0.2829	0.3127	0.4043
2	2	2	2	15	2	0.2790	0.3084	0.4124
2	2	2	2	2	30	0.3642	0.4072	0.2329

Table 4. Bayesian A-optimal design for Gamma regression model [Support points (0, 0), (1, 0), (0, 1) and $\beta_0 \sim G(\alpha_1,\delta_1),\beta_1 \sim G(\alpha_2,\delta_2),\beta_2 \sim G(\alpha_3,\delta_3)$]

α_1	δ_1	α_2	δ_2	α_3	δ_3	w_1	w_2	w_3
2	2	2	2	2	2	0.3217	0.3391	0.3391
2	5	2	2	2	2	0.3995	0.3002	0.3002
2	10	2	2	2	2	0.4310	0.2844	0.2844
3	15	3	3	3	3	0.4279	0.2860	0.2860
3	20	3	3	3	3	0.4367	0.2816	0.2816
5	2	3	3	3	3	0.3195	0.3402	0.3402
10	2	3	3	3	3	0.3761	0.3119	0.3119
15	3	2	2	2	2	0.4438	0.2780	0. 2780
20	3	3	3	3	3	0.4303	0.2848	0.2848
2	2	2	5	2	2	0.2543	0.4776	0.2680
2	2	2	10	2	2	0.1867	0.6163	0.1968
2	2	2	20	2	2	0.1215	0.7503	0.1281
2	2	5	2	2	2	0.2619	0.4619	0.2760
2	2	10	2	2	2	0.2008	0.5890	0.2190
2	2	20	2	2	2	0.1359	0.7207	0.1432
2	2	2	2	2	5	0.2543	0.2680	0.4776
2	2	2	2	5	2	0.2619	0.2760	0.4619
2	2	2	2	2	10	0.1867	0.1968	0.6163
2	2	2	2	10	2	0.2000	0.2109	0.5890
2	2	2	2	2	20	0.1215	0.1281	0.7503

Table 5. Bayesian A-optimal design for Gamma regression model [Support points (0, 0), (1, 0), (1, 1) and $\beta_0 \sim U(a,b),\beta_1 \sim U(c,d),\beta_2 \sim U(e,f)$]

a	b	c	d	e	f	w_1	w_2	w_3
0	1	0	1	0	1	0.2080	0.3891	0.4028
0	10	0	1	0	1	0.3455	0.3721	0.2823
0	20	0	1	0	1	0.3572	0.3708	0.2319
0	50	0	1	0	1	0.3644	0.3699	0.2655
0	100	0	1	0	1	0.3669	0.3696	0.2633
0	500	0	1	0	1	0.3689	0.3694	0.2616
0	1000	0	1	0	1	0.3691	0.3694	0.2614
0	1	0	10	0	1	0.0501	0.5401	0.4097
0	1	0	20	0	1	0.0270	0.5613	0.4116
0	1	0	50	0	1	0.0113	0.5755	0.4103
0	1	0	100	0	1	0.0057	0.5866	0.4136
0	1	0	500	0	1	0.0011	0.5847	0.4140
0	1	0	1000	0	1	0.0005	0.5852	0.4141
0	1	0	1	0	10	0.0905	0.1694	0.7399
0	1	0	1	0	20	0.0552	0.1033	0.8413
0	1	0	1	0	50	0.0254	0.0476	0.9269
0	1	0	1	0	100	0.0133	0.0250	0.9615
0	1	0	1	0	500	0.0027	0.0052	0.9919
0	1	0	1	0	1000	0.0014	0.0026	0.9959
20	50	60	90	100	120	0.1191	0.3657	0.5150

Table 6. Bayesian A-optimal design for Gamma regression model [Support points (0, 0), (1, 0), (1, 1) and $\beta_0 \sim N(\mu_1,\sigma_1),\beta_1 \sim N(\mu_2,\sigma_2),\beta_2 \sim N(\mu_3,\sigma_3)$]

μ_1	σ_1	μ_2	σ_2	μ_3	σ_3	w_1	w_2	w_3
1	5	2	10	3	15	0.1668	0.3787	0.4544
2	4	3	5	4	6	0.2081	0.3781	0.4137
2	5	4	8	3	5	0.2036	0.4228	0.3734
3	10	5	15	10	12	0.2084	0.3938	0.3976
5	1	10	2	15	3	0.1224	0.3641	0.5133
4	2	5	3	6	4	0.1758	0.3811	0.4430
5	2	8	4	5	3	0.1661	0.4240	0.4098
10	3	15	5	12	10	0.1643	0.4041	0.4315
10	20	15	50	20	80	0.1424	0.3782	0.4793
15	25	20	40	30	60	0.1836	0.3699	0.4464
30	45	50	60	65	75	0.1859	0.3771	0.4368
40	50	100	200	250	300	0.0927	0.3609	0.5463
50	40	120	150	150	180	0.1111	0.3996	0.4892
10	5	20	10	3	15	0.1596	0.4572	0.3830
2	14	3	15	4	16	0.1183	0.4316	0.4499
3	100	5	150	10	120	0.2304	0.4156	0.3539
12	15	18	30	40	45	0.1504	0.3525	0.4969
-60	65	-62	68	-70	75	0.2196	0.3825	0.3978
-12	15	-18	20	-15	20	0.1974	0.4013	0.4013
-5	10	15	20	-25	30	0.1742	0.3816	0.4441

Table 7. Bayesian A-optimal design for Gamma regression model [Support points (0, 0), (1, 0), (1, 1) and $\beta_0 \sim B(a_1,b_1),\beta_1 \sim B(a_2,b_2),\beta_2 \sim B(a_3,b_3)$]

a_1	b_1	a_2	b_2	a_3	b_3	w_1	w_2	w_3
5	5	5	5	5	5	0.1992	0.3900	0.4107
5	10	5	5	5	5	0.1631	0.3945	0.4422
5	50	5	5	5	5	0.0659	0.4084	0.5256
5	100	5	5	5	5	0.0377	0.4129	0.5493
10	5	5	5	5	5	0.2219	0.3875	0.3904
50	5	5	5	5	5	0.2468	0.3845	0.3685
100	5	5	5	5	5	0.2505	0.3841	0.3653
5	5	5	10	5	5	0.2235	0.3660	0.4103
5	5	5	50	5	5	0.2720	0.3182	0.4096

5	5	5	100	5	5	0.2827	0.3077	0.4094
5	5	10	5	5	5	0.1803	0.4083	0.4113
5	5	50	5	5	5	0.1583	0.4297	0.4118
5	5	100	5	5	5	0.1549	0.4331	0.4119
5	5	5	5	5	10	0.2086	0.4084	0.3828
5	5	5	5	5	20	0.2168	0.4245	0.3585
5	5	5	5	5	50	0.2240	04386.	0.3373
5	5	5	5	5	100	0.2269	0.4443	0.3286
5	5	5	5	10	5	0.1908	0.3737	0.4353
5	5	5	5	20	5	0.1846	0.3616	0.4537
5	5	5	5	100	5	0.1780	0.3485	0.4734

Table 8. Bayesian A-optimal design for Gamma regression model [Support points (0, 0), (1, 0), (1, 1) and $\beta_0 \sim G(\alpha_1, \delta_1), \beta_1 \sim G(\alpha_2, \delta_2), \beta_2 \sim G(\alpha_3, \delta_3)$]

α_1	δ_1	α_2	δ_2	α_3	δ_3	w_1	w_2	w_3
2	2	2	2	2	2	0.2129	0.3887	0.3983
2	5	2	2	2	2	0.2905	0.3781	0.3313
2	10	2	2	2	2	0.3270	0.3740	0.2987
2	20	2	2	2	2	0.3477	0.3718	0.2803
5	2	2	2	2	2	0.2789	0.3804	0.3459
10	2	2	2	2	2	0.3156	0.3759	0.3084
20	2	2	2	2	2	0.3398	0.3729	0.2871
2	2	2	5	2	2	0.1407	0.4579	0.4012
2	2	2	10	2	2	0.0886	0.5066	0.4046
2	2	2	20	2	2	0.0506	0.5412	0.4081
2	2	5	2	2	2	0.1471	0.4496	0.4031
2	2	10	2	2	2	0.0972	0.4959	0.4068
2	2	20	2	2	2	0.0579	0.5322	0.4098
2	2	2	2	2	5	0.1761	0.3216	0.5021
2	2	2	2	2	10	0.1353	0.2470	0.6176
2	2	2	2	2	20	0.0918	0.1676	0.7404
2	2	2	2	5	2	0.1797	0.3281	0.4921
2	2	2	2	10	2	0.1426	0.2604	0.5968
2	2	2	2	20	2	0.1010	0.1844	0.7144
2	2	2	2	30	2	0.0782	0.1428	0.7789

Table 9. Bayesian A-optimal design for Gamma regression model [Support points (0, 0), (0, 1), (1, 1) and $\beta_0 \sim U(a, b), \beta_1 \sim U(c, d), \beta_2 \sim U(e, f)$]

a	b	c	d	e	f	w_1	w_2	w_3
0	1	0	1	0	1	0.2080	0.3891	0.4028
0	10	0	1	0	1	0.3455	0.3721	0.2823
0	20	0	1	0	1	0.3572	0.3708	0.2719
0	40	0	1	0	1	0.3632	0.3701	0.2666
0	60	0	1	0	1	0.3653	0.3698	0.2648
0	120	0	1	0	1	0.3673	0.3696	0.2630
0	200	0	1	0	1	0.3681	0.3695	0.2622
0	1	0	10	0	1	0.0905	0.1694	0.7399
0	1	0	20	0	1	0.0552	0.1033	0.8413
0	1	0	40	0	1	0.0310	0.0580	0.9190
0	1	0	60	0	1	0.0215	0.0403	0.9380
0	1	0	120	0	1	0.0112	0.0210	0.9676
0	1	0	200	0	1	0.0068	0.0128	0.9802
0	1	0	1	0	10	0.0501	0.5401	0.4097
0	1	0	1	0	20	0.0270	0.5613	0.4116
0	1	0	1	0	40	0.0140	0.5731	0.4128
0	1	0	1	0	60	0.0095	0.5772	0.4132
0	1	0	1	0	120	0.0048	0.5814	0.4137
0	1	0	1	0	200	0.0029	0.5831	0.4139
10	20	30	40	50	60	0.0956	0.4389	0.4653

Table 10. Bayesian A-optimal design for Gamma regression model [Support points (0, 0), (0, 1), (1, 1) and $\beta_0 \sim N(\mu_1, \sigma_1), \beta_1 \sim N(\mu_2, \sigma_2), \beta_2 \sim N(\mu_3, \sigma_3)$]

μ_1	σ_1	μ_2	σ_2	μ_3	σ_3	w_1	w_2	w_3
1	5	5	2	10	15	0.1264	0.4776	0.3959
5	55	54	27	120	159	0.1256	0.4768	0.3975
2	10	5	20	10	30	0.1395	0.4628	0.3976
5	8	10	15	20	25	0.0212	0.5717	0.4069
8	13	15	18	33	35	0.1272	0.4621	0.4106
10	15	12	19	20	23	0.1865	0.4208	0.3926
20	25	22	28	40	45	0.1771	0.4374	0.3853
25	5	50	10	100	20	0.0920	0.4573	0.4505
50	15	100	75	35	25	0.1833	0.3156	0.5010
100	120	150	200	300	350	0.1311	0.4576	0.4111
120	100	200	150	350	300	0.1242	0.4505	0.4251
220	250	270	290	300	330	0.2005	0.4002	0.3992
50	175	120	150	180	200	0.2111	0.4078	0.3809
425	450	350	360	370	390	0.2418	0.3880	0.3701
-20	50	-50	120	-12	50	0.2191	0.3158	0.4650
-50	150	-60	150	-18	35	0.3140	0.3344	0.3514
-65	250	-70	190	-19	60	0.3093	0.3595	0.3310
-70	120	-80	200	-25	70	0.2675	0.3291	0.4083
10	50	100	500	150	400	0.0512	0.4479	0.5008
25	50	75	100	125	150	0.1164	0.4533	0.4303

Table 11. Bayesian A-optimal design for Gamma regression model [Support points (0, 0), (0, 1), (1, 1) and $\beta_0 \sim B(a_1,b_1), \beta_1 \sim B(a_2,b_2), \beta_2 \sim B(a_3,b_3)$]

a_1	b_1	a_2	b_2	a_3	b_3	w_1	w_2	w_3
3	3	3	3	3	3	0.2013	0.3898	0.4088
3	6	3	3	3	3	0.1659	0.3942	0.4397
3	10	3	3	3	3	0.1341	0.3987	0.4671
3	20	3	3	3	3	0.0903	0.4054	0.5041
3	50	3	3	3	3	0.0454	0.4132	0.5412
6	3	3	3	3	3	0.2223	0.3876	0.3899
10	3	3	3	3	3	0.2332	0.3864	0.3803
20	3	3	3	3	3	0.2427	0.3852	0.3719
50	3	3	3	3	3	0.2492	0.3844	03663.
3	3	3	10	3	3	0.2169	0.4201	0.3629
3	3	3	50	3	3	0.2283	0.4422	0.3293
3	3	3	100	3	3	0.2302	0.4458	0.3238
3	3	10	3	3	3	0.1883	0.3648	0.4468
3	3	50	3	3	3	0.1807	0.3500	0.4691
3	3	100	3	3	3	0.1796	0.3478	0.4725
3	3	3	3	3	10	0.2438	0.3481	0.4079
3	3	3	3	3	50	0.2821	0.3107	0.4070
3	3	3	3	50	3	0.1582	0.4310	0.4107
3	3	3	3	10	3	0.1729	0.4168	0.4102
3	3	3	3	100	3	0.1561	0.4331	0.4107

Table 12. Bayesian A-optimal design for Gamma regression model [Support points (0, 0), (0, 1), (1, 1) and $\beta_0 \sim G(\alpha_1,\delta_1), \beta_1 \sim G(\alpha_2,\delta_2), \beta_2 \sim G(\alpha_3,\delta_3)$]

α_1	δ_1	α_2	δ_2	α_3	δ_3	w_1	w_2	w_3
3	3	3	3	3	3	0.2080	0.3891	0.4028
3	5	3	3	3	3	0.2545	0.3827	0.3627
3	10	3	3	3	3	0.3036	0.3768	0.3195
3	20	3	3	3	3	0.3343	0.3734	0.2922
5	3	3	3	3	3	0.2479	0.3841	0.3678
10	3	3	3	3	3	0.2942	0.3785	0.3272
20	3	3	3	3	3	0.3267	0.3745	0.2986
3	3	3	5	3	3	0.1906	0.3566	0.4527
3	3	3	10	3	3	0.1567	0.2932	0.5499
3	3	3	20	3	3	0.1149	0.2150	0.6699
3	3	5	3	3	3	0.1917	0.3586	0.4496
3	3	10	3	3	3	0.1603	0.2999	0.5397
3	3	20	3	3	3	0.1207	0.2259	0.6532
3	3	3	3	3	5	0.1700	0.4261	0.4038
3	3	3	3	3	10	0.1156	0.4785	0.4057
3	3	3	3	3	20	0.0700	0.5216	0.4083
3	3	3	3	5	3	0.1725	0.4227	0.4047
3	3	3	3	10	3	0.1210	0.4714	0.4075
3	3	3	3	20	3	0.0758	0.5141	0.4100
3	3	3	3	40	3	0.0275	0.5598	0.4126

Table 13. Bayesian A-optimal design for Gamma regression model [Support points (1, 0), (0, 1), (1, 1) and $\beta_0 \sim U(a,b), \beta_1 \sim U(c,d), \beta_2 \sim U(e,f)$]

a	b	c	d	e	f	w_1	w_2	w_3
0	1	0	1	0	1	0.2636	0.2636	0.4726
0	10	0	1	0	1	0.3017	0.3017	0.3964
0	20	0	1	0	1	0.3058	0.3058	0.3883
0	50	0	1	0	1	0.3083	0.3083	0.3832
0	100	0	1	0	1	0.3092	0.3092	0.3815
0	500	0	1	0	1	0.3099	0.3099	0.3801
0	1000	0	1	0	1	0.3100	0.3100	0.3799
0	1	0	10	0	1	0.4019	0.0698	0.5281
0	1	0	20	0	1	0.4236	0.0381	0.5381
0	1	0	50	0	1	0.4386	0.0161	0.5452
0	1	0	100	0	1	0.4439	0.0082	0.5478
0	1	0	500	0	1	0.4483	0.0016	0.5499
0	1	0	1000	0	1	0.4489	0.0008	0.5502
0	1	0	1	0	10	0.0698	0.4019	0.5281
0	1	0	1	0	20	0.0381	0.4236	0.5381
0	1	0	1	0	50	0.0161	0.4386	0.5452
0	1	0	1	0	100	0.0082	0.4439	0.5478
0	1	0	1	0	500	0.0016	0.4483	0.5499
0	1	0	1	0	1000	0.0008	0.4489	0.5502
30	50	60	90	100	1500	0.2004	0.2877	0.5117

Table 14. Bayesian A-optimal design for Gamma regression model [Support points (1, 0), (0, 1), (1, 1) and $\beta_0 \sim N(\mu_1,\sigma_1), \beta_1 \sim N(\mu_2,\sigma_2), \beta_2 \sim N(\mu_3,\sigma_3)$]

μ_1	σ_1	μ_2	σ_2	μ_3	σ_3	w_1	w_2	w_3
2	5	3	10	4	20	0.1917	0.3361	0.4720
3	15	2	5	4	30	0.1767	0.3652	0.4580
6	40	7	50	8	90	0.2159	0.3288	0.4551
20	30	40	50	60	70	0.2209	0.2917	0.4873
80	90	100	120	130	140	0.2946	0.2798	0.4754
25	50	75	100	125	150	0.2065	0.3000	0.4934
200	250	270	290	300	350	0.2522	0.2744	0.4732

5	2	10	3	20	4	0.1832	0.3016	0.5151
15	3	5	2	30	4	0.1597	0.3559	0.4842
40	6	50	7	90	7	0.2048	0.2950	0.5001
-10	15	-7	30	-5	10	0.3366	0.2099	0.4533
-20	30	-21	60	-10	20	0.3394	0.2025	0.4580
-20	45	-14	180	-15	30	0.3842	0.1312	0.4845
-25	60	-28	90	-20	40	0.3232	0.2281	0.4486
-20	70	-29	180	-25	50	0.3577	0.1743	0.4679
15	30	60	80	90	180	0.1773	0.2284	0.4942
200	250	-20	100	10	15	0.3070	0.3098	0.3830
250	240	-40	50	20	40	0.2932	0.3301	0.3766
240	100	-60	80	40	60	0.2615	0.3591	0.3792
300	350	-90	100	50	65	0.2852	0.3388	0.3758

Table 15. Bayesian A-optimal design for Gamma regression model [Support points (1, 0), (0, 1), (1, 1) and $\beta_0 \sim B(a_1,b_1)$, $\beta_1 \sim B(a_2,b_2)$, $\beta_2 \sim B(a_3,b_3)$]

a_1	b_1	a_2	b_2	a_3	b_3	w_1	w_2	w_3
4	4	4	4	4	4	0.2617	0.2617	0.4765
4	10	4	4	4	4	0.2512	0.2512	0.4975
4	20	4	4	4	4	0.2431	0.2431	0.5137
4	50	4	4	4	4	0.2352	0.2352	0.5294
4	100	4	4	4	4	0.2318	0.2318	0.5362
4	500	4	4	4	4	0.2287	0.2287	0.5424
10	4	4	4	4	4	0.2684	0.2684	0.4631
20	4	4	4	4	4	0.2714	0.2714	0.4570
100	4	4	4	4	4	0.2743	0.2743	0.4512
500	4	4	4	4	4	0.2749	0.2749	0.4500
4	4	4	10	4	4	0.2355	0.2981	0.4663
4	4	4	50	4	4	0.2013	0.3454	0.4531
4	4	4	100	4	4	0.1947	0.3547	0.4504
4	4	4	500	4	4	0.1889	0.3630	0.4480
4	4	10	4	4	4	0.2809	0.2344	0.4846
4	4	50	4	4	4	0.2966	0.2122	0.4910
4	4	4	4	4	10	0.2981	0.2355	0.4663
4	4	4	4	4	50	0.3454	0.2013	0.4531
4	4	4	4	10	4	0.2344	0.2809	0.4846
4	4	4	4	50	4	0.2122	0.2966	0.4910

Table 16. Bayesian A-optimal design for Gamma regression model [Support points (1, 0), (0, 1), (1, 1) and $\beta_0 \sim G(\alpha_1,\delta_1)$, $\beta_1 \sim G(\alpha_2,\delta_2)$, $\beta_2 \sim G(\alpha_3,\delta_3)$]

α_1	δ_1	α_2	δ_2	α_3	δ_3	w_1	w_2	w_3
4	4	4	4	4	4	0.2629	0.2629	0.4740
4	10	4	4	4	4	0.2822	0.2822	0.4354
4	15	4	4	4	4	0.2859	0.2859	0.4208
4	30	4	4	4	4	0.2987	0.2987	0.4025
4	50	4	4	4	4	0.3029	0.3029	0.3940
4	10	4	10	4	4	0.3168	0.2240	0.4591
4	4	4	15	4	4	0.3488	0.1442	0.5069
4	4	4	30	4	4	0.3882	0.0885	0.5232
4	4	4	50	4	4	0.4093	0.0583	0.5323
4	4	10	4	4	4	0.3176	0.1859	0.4964
4	4	15	4	4	4	0.3434	0.1495	0.5070
4	4	30	4	4	4	0.3827	0.0941	0.5231
4	4	50	4	4	4	0.4048	0.0630	0.5321
4	4	4	4	4	10	0.1819	0.3217	0.4963
4	4	4	4	4	15	0.1442	0.3488	0.5069
4	4	4	4	4	30	0.0885	0.3882	0.5232
4	4	4	4	4	50	0.0583	0.4093	0.5323
4	4	4	4	10	4	0.1859	0.3176	0.4964
4	4	4	4	15	4	0.1495	0.3434	0.5070
4	4	4	4	30	4	0.0941	0.3827	0.5231

Table 17. Bayesian A-optimal design for Poisson regression model [Support points (0, 0), (1, 0), (0, 1) and $\beta_0 \sim G(\alpha_1,\delta_1)$, $\beta_1 \sim G(\alpha_2,\delta_2)$, $\beta_2 \sim G(\alpha_3,\delta_3)$]

α_1	δ_1	α_2	δ_2	α_3	δ_3	w_1	w_2	w_3
1	2	3	4	5	6	0.9468	0.0488	0.0042
2	2	2	2	2	2	0.7220	0.1359	0.1389
2	10	3	15	4	20	0.9897	0.0089	0.0012
2	20	3	3	3	3	0.8738	0.0630	0.0630
2	20	3	3	3	10	0.9757	0.0088	0.0151
50	20	2	3	5	10	0.8727	0.1259	0.0012
1	5	3	5	2	3	0.8448	0.0331	0.1219
10	20	4	4	4	4	0.9558	0.0220	0.0220
3	3	3	15	3	3	0.9249	0.0083	0.0667
3	3	3	50	3	3	0.9313	0.0014	0.0672
3	3	3	75	3	3	0.9319	0.0008	0.0672
3	3	15	3	3	3	0.9326	0.0001	0.0673
3	3	5	3	3	3	0.9172	0.0165	0.0661
3	3	7	4	3	3	0.9308	0.0019	0.0671
2	2	2	2	2	3	0.7480	0.1439	0.1079
2	2	2	2	2	5	0.7759	0.1493	0.0746
2	2	2	2	2	10	0.8032	0.1545	0.0421
2	2	2	2	3	2	0.7671	0.1476	0.0852
2	2	2	2	5	2	0.8133	0.1565	0.0301
2	2	2	2	10	2	0.8369	0.1610	0.0019

Table 18. Bayesian A-optimal design for Poisson regression model [Support points (0, 0), (1, 0), (1, 1) and $\beta_0 \sim G(\alpha_1,\delta_1), \beta_1 \sim G(\alpha_2,\delta_2), \beta_2 \sim G(\alpha_3,\delta_3)$]

α_1	δ_1	α_2	δ_2	α_3	δ_3	w_1	w_2	w_3
2	2	2	2	2	2	0.7082	0.2360	0.0556
5	2	3	2	2	2	0.8078	0.1554	0.0366
2	10	4	6	5	10	0.9799	0.0199	0.0003
2	15	3	10	10	15	0.9733	0.0266	0.0001
2	3	5	5	5	5	0.9886	0.0112	0.0008
5	3	5	2	5	1	0.9326	0.0598	0.0074
10	2	5	3	6	4	0.9695	0.0302	0.0001
15	2	3	10	4	10	0.9731	0.0266	0.0001
2	3	4	5	8	9	0.8225	0.1645	0.0129
2	3	2	4	2	3	0.8094	0.1618	0.0286
3	5	2	4	2	5	0.8172	0.1634	0.0192
3	3	2	15	3	3	0.9363	0.0585	0.0051
3	5	2	20	3	10	0.9537	0.0454	0.0008
10	5	5	2	7	3	0.9394	0.0602	0.0003
2	2	10	2	2	2	0.9949	0.0040	0.0009
5	2	5	5	5	5	0.9886	0.0112	0.0008
3	4	2	6	7	10	0.8749	0.1249	0.0002
3	5	2	6	2	5	0.8622	0.1231	0.0145
3	10	7	9	5	2	0.9996	0.0003	0.0001
1	3	2	5	3	6	0.8524	0.1420	0.0054

Table 19. Bayesian A-optimal design for Poisson regression model [Support points (0, 0), (0, 1), (1, 1) and $\beta_0 \sim G(\alpha_1,\delta_1), \beta_1 \sim G(\alpha_2,\delta_2), \beta_2 \sim G(\alpha_3,\delta_3)$]

α_1	δ_1	α_2	δ_2	α_3	δ_3	w_1	w_2	w_3
3	3	3	3	3	3	0.8802	0.1100	0.0097
3	3	5	3	3	3	0.8867	0.1108	0.0024
3	3	3	3	10	3	0.9989	0.0009	0.0008
2	3	2	5	3	9	0.9658	0.0305	0.0035
3	2	5	2	9	3	0.9979	0.0019	0.0008
1	5	2	10	3	15	0.9836	0.0153	0.0009
2	4	2	6	2	8	0.8910	0.0989	0.0100
2	5	2	10	2	15	0.9376	0.0586	0.0037
2	5	3	5	6	6	0.9969	0.0029	0.0001
1	1	1	1	1	2	0.5358	0.3094	0.1547
1	1	1	2	1	1	0.5010	0.3543	0.1446
1	2	1	1	1	1	0.4852	0.3431	0.1715
1	2	1	2	1	2	0.5515	0.3184	0.1300
1	5	1	5	1	5	0.6552	0.2675	0.0772
2	5	2	5	2	5	0.8429	0.1404	0.0165
1	5	1	7	2	5	0.8275	0.1379	0.0344
1	6	1	9	2	7	0.8673	0.1048	0.0242
3	1	2	3	3	2	0.8153	0.1569	0.0277
5	1	2	1	7	1	0.8931	0.0789	0.0279
1	9	1	10	1	15	0.7672	0.1918	0.0408

Table 20. Bayesian A-optimal design for Poisson regression model [Support points (1, 0), (0, 1), (1, 1) and $\beta_0 \sim G(\alpha_1,\delta_1), \beta_1 \sim G(\alpha_2,\delta_2), \beta_2 \sim G(\alpha_3,\delta_3)$]

α_1	δ_1	α_2	δ_2	α_3	δ_3	w_1	w_2	w_3
1	1	1	1	1	1	0.3489	0.3489	0.3021
1	5	1	3	1	10	0.5070	0.3057	0.1872
1	7	1	6	1	12	0.4822	0.3538	0.1638
1	9	1	10	1	15	0.4683	0.3883	0.1433
5	2	6	2	10	2	0.8959	0.0995	0.0045
2	3	4	5	6	7	0.9322	0.0655	0.0022
2	1	3	1	10	1	0.8875	0.0784	0.0339
5	1	10	1	20	1	0.9685	0.0302	0.0011
2	5	2	7	2	9	0.5201	0.4161	0.0637
2	4	2	6	2	8	0.5225	0.4063	0.0711
2	3	2	6	2	9	0.5487	0.3840	0.0672
4	4	4	4	4	1	0.1323	0.8271	0.0405
1	4	4	1	4	4	0.8271	0.1323	0.0405
4	5	4	7	4	9	0.6052	0.3873	0.0074
1	15	2	10	3	15	0.8396	0.1443	0.0160
1	2	1	5	1	3	0.3524	0.4316	0.2158
1	2	2	4	3	5	0.7024	0.2389	0.0585
2	2	2	4	6	5	0.9719.	0.0224	0.0055
4	1	10	2	9	3	0.6770	0.3213	0.0016
1	2	3	4	5	6	0.9126	0.0787	0.0086

Table 21. A-efficiency forGamma regression model [Support points (0, 0), (0, 1), (1, 1) and $\beta_0 \sim U(a,b), \beta_1 \sim U(c,d), \beta_2 \sim U(e,f)$]

Particulars	Design 1	Design 2	Design 3	Design 4	Design 5
β_0	1	2	4	6	10
β_1	3	5	6	15	20
β_2	4	7	15	20	30
w_1	0.3731	0.3801	0.3878	0.3812	0.3881
w_2	0.3046	0.3001	0.2815	0.3010	0.2949
w_3	0.3221	0.3196	0.3305	0.3176	0.3169
Tr($M^{-1}(\xi)$)	**138.184**	**448.291**	**1571.3**	**3876.9**	**8873.79**
a,b (prior)	(0,5)	(1,3)	(0,9)	(5,12)	(8,18)

c,d (prior)	(0,5)	(2,10)	(4,10)	(10,25)	(18,28)
e,f (prior)	(0,5)	(5,10)	(10,20)	(15,25)	(28,38)
w_1	0.3164	0.1670	0.2205	0.2153	0.2186
w_2	0.3417	0.3868	0.2920	0.3761	0.3434
w_3	0.3417	0.4461	0.4873	0.4085	0.4378
Tr($M^{-1}(\xi)$)	**129.47**	**380.11**	**1300.97**	**3329.02**	**7647.86**
Efficiency	1.02	1.05	1.06	1.05	1.05

Table 22. A- efficiency for Gamma regression model [Support points (0, 0), (0, 1), (1, 1) and $\beta_0 \sim N(\mu_1,\sigma_1), \beta_1 \sim N(\mu_2,\sigma_2), \beta_2 \sim N(\mu_3,\sigma_3)$]

Particulars	Design 1	Design 2	Design 3	Design 4	Design 5
β_0	3	5	7	10	25
β_1	5	10	12	22	40
β_2	9	20	30	45	80
w_1	0.3914	0.3812	0.3821	0.3764	0.3906
w_2	0.2888	0.2896	0.2832	0.2906	0.2864
w_3	0.3196	0.3291	0.3345	0.3328	0.3229
Tr($M^{-1}(\xi)$)	**741.153**	**2872.8**	**5752.11**	**13410.3**	**53696.1**
μ_1,σ_1 (prior)	(0,6)	(-1,10)	(5,15)	(-20,20)	(-10,40)
μ_2,σ_2 (prior)	(2,8)	(-2,15)	(10,20)	(10,30)	(-5,45)
μ_3,σ_3 (prior)	(5,10)	(-3,25)	(20,35)	(-50,50)	(50,100)
w_1	0.3122	0.2767	0.2682	0.2803	0.2875
w_2	0.3064	0.2905	0.2855	0.2141	0.2498
w_3	0.3812	0.4327	0.4461	0.5054	0.4626
Tr($M^{-1}(\xi)$)	**673.115**	**2490**	**4881.37**	**11838.5**	**47268**
Efficiency	1.03	1.04	1.05	1.04	1.04

Table 23. A-efficiency for Gamma regression model [Support points (0, 0), (0, 1), (1, 1) and $\beta_0 \sim G(\alpha_1,\delta_1), \beta_1 \sim G(\alpha_2,\delta_2), \beta_2 \sim G(\alpha_3,\delta_3)$]

Particulars	Design 1	Design 2	Design 3	Design 4	Design 5
β_0	2	3	5	6	12
β_1	3	8	10	15	34
β_2	11	12	18	25	40
w_1	0.3776	0.3756	0.3838	0.3759	0.3786
w_2	0.2741	0.3000	0.2916	0.2968	0.3059
w_3	0.3481	0.3242	0.3245	0.3272	0.3154
Tr($M^{-1}(\xi)$)	**608.48**	**1169.23**	**2597.22**	**4710.2**	**16631.6**
α_1,δ_1 (prior)	(1,5)	(10,2)	(5,2)	(2,2)	(10,15)
α_2,δ_2 (prior)	(2,6)	(3,3)	(3,3)	(2,2)	(30,40)
α_3,δ_3 (prior)	(5,15)	(3,3)	(3,3)	(2,2)	(35,50)
w_1	0.1031	0.3761	0.3195	0.0769	0.0764
w_2	0.1653	0.3119	0.3402	0.2676	0.3839
w_3	0.7314	0.3119	0.3402	0.6554	0.5396
Tr($M^{-1}(\xi)$)	**498.696**	**1181.12**	**2451.09**	**4470.19**	**16177.4**
Efficiency	1.06	0.99	1.01	1.01	1.01

Table 24. Design efficiency for Gamma regression model [Support points (0, 0), (0, 1), (1, 1) and $\beta_0 \sim B(a_1,b_1), \beta_1 \sim B(a_2,b_2), \beta_2 \sim B(a_3,b_3)$]

Particulars	Design 1	Design 2	Design 3	Design 4	Design 5
β_0	0.15	0.31	0.1	0.24	0.12
β_1	0.3	0.44	0.26	0.52	0.35
β_2	0.45	0.7	0.32	0.92	0.95
w_1	0.3881	0.4006	0.3814	0.3807	0.3558
w_2	0.2949	0.2885	0.3033	0.2932	0.2890
w_3	0.3169	0.3108	0.3152	0.3259	0.3550
Tr($M^{-1}(\xi)$)	**1.9966**	**5.9515**	**1.0656**	**6.5527**	**4.1108**
a_1,b_1 (prior)	(1,3)	(2,5)	(4,10)	(15,20)	(25,30)
a_2,b_2 (prior)	(2,5)	(3,8)	(3,9)	(15,30)	(35,40)
a_3,b_3 (prior)	(3,6)	(5,10)	(4,20)	(15,40)	(45,60)
w_1	0.3164	0.3227	0.3225	0.3293	0.3006
w_2	0.3417	0.3386	0.3387	0.3353	0.3496
w_3	0.3417	0.3386	0.3387	0.3353	0.3496
Tr($M^{-1}(\xi)$)	**1.8595**	**5.5673**	**0.9964**	**6.2605**	**4.0504**
Efficiency	1.02	1.02	1.02	1.02	1.01

Appendix II. Information matrix for Gamma regression model

Case 1. When the support points are $\boldsymbol{x}_1^*$, $\boldsymbol{x}_2^*$, and $\boldsymbol{x}_3^*$. The inverse of the information matrix is given by

$$\boldsymbol{M}^{-1}(\xi)=\begin{bmatrix} a_{11}^+ & a_{12}^+ & a_{13}^+ \\ a_{21}^+ & a_{22}^+ & a_{23}^+ \\ a_{31}^+ & a_{32}^+ & a_{33}^+ \end{bmatrix}$$

where $a_{11}^+=a_{23}^+=a_{32}^+=\dfrac{\beta_0^2}{w_1}$,

$$a_{12}^+=a_{13}^+=a_{21}^+=a_{31}^+=\frac{-\beta_0^2}{w_1},$$

$$a_{22}^+=\frac{(w_1+w_2)\beta_0^2+2w_1\beta_0\beta_1+w_1\beta_1^2}{w_1w_2},$$

$$a_{33}^+=\frac{(w_1+w_3)\beta_0^2+2w_1\beta_0\beta_2+w_1\beta_2^2}{w_1w_3}, \text{ and}$$

$$\mathrm{Tr}\left(\boldsymbol{M}^{-1}(\xi)\right)=\frac{3w_2w_3+w_1(w_1+w_3)\beta_0^2+2w_1\beta_0(w_3\beta_1+w_2\beta_2)+w_1\left(w_3\beta_1^2+w_2\beta_2^2\right)}{w_1w_2w_3}.$$

Case 2. When the support points are $\boldsymbol{x}_1^*$, $\boldsymbol{x}_2^*$, and $\boldsymbol{x}_4^*$. The inverse of the information matrix is given by

$$\boldsymbol{M}^{-1}(\xi)=\begin{bmatrix} b_{11}^+ & b_{12}^+ & b_{13}^+ \\ b_{21}^+ & b_{22}^+ & b_{23}^+ \\ b_{31}^+ & b_{32}^+ & b_{33}^+ \end{bmatrix}$$

where $b_{11}^+=\dfrac{\beta_0^2}{w_1}$, $a_{12}^+=a_{21}^+=\dfrac{-\beta_0^2}{w_1}$, $b_{13}^+=b_{31}^+=0$,

$$b_{23}^+=a_{32}^+=\frac{-(\beta_0+\beta_1)^2}{w_2},$$

$$b_{22}^+=b_{33}^+=\frac{(w_2+w_3)(\beta_0+\beta_1)^2+2w_2(\beta_0+\beta_1)\beta_2+w_2\beta_2^2}{w_2w_3}, \text{ and}$$

$$\mathrm{Tr}\left(\boldsymbol{M}^{-1}(\xi)\right)=\frac{(w_1w_2+2(w_1+w_2)w_3)\beta_0^2+2w_1\beta_0((w_2+2w_3)\beta_1+w_2\beta_2)+w_1((w_2+2w_3)\beta_1^2+2w_2\beta_1\beta_2+w_2\beta_2^2)}{w_1w_2w_3}.$$

Case 3. When the support points are $\boldsymbol{x}_1^*, \boldsymbol{x}_3^*, and\ \boldsymbol{x}_4^*$. The inverse of the information matrix is given by

$$\boldsymbol{M}^{-1}(\xi)=\begin{bmatrix} c_{11}^+ & c_{12}^+ & c_{13}^+ \\ c_{21}^+ & c_{22}^+ & c_{23}^+ \\ c_{31}^+ & c_{32}^+ & c_{33}^+ \end{bmatrix}$$

where $c_{11}^+=\dfrac{\beta_0^2}{w_1}$, $c_{12}^+=c_{21}^+=0$, $c_{13}^+=c_{31}^+=\dfrac{-\beta_0^2}{w_1}$,

$$c_{23}^+=c_{32}^+=\frac{-(\beta_0+\beta_2)^2}{w_2},$$

$$c_{22}^+=\frac{(w_2+w_3)\beta_0^2+w_2\beta_1^2+2w_2\beta_1\beta_2+(w_2+w_3)\beta_2^2+2\beta_0(w_2\beta_1+(w_2+w_3)\beta_2)}{w_2w_3},$$

$$c_{33}^+=\frac{(w_1+w_2)\beta_0^2+2w_1\beta_0\beta_2+w_1\beta_2^2}{w_1w_2}, \text{ and}$$

$$\mathrm{Tr}\left(\boldsymbol{M}^{-1}(\xi)\right)=\frac{(w_1w_2+2(w_1+w_2)w_3)\beta_0^2+2w_1\beta_0(w_2\beta_1+(w_2+2w_3)\beta_2)+w_1\left(w_2\beta_1^2+2w_2\beta_1\beta_2+(w_2+2w_3)\beta_2^2\right)}{w_1w_2w_3}.$$

Case 4. When the support points are $\boldsymbol{x}_2^*$, $\boldsymbol{x}_3^*$, and $\boldsymbol{x}_4^*$. The inverse of the information matrix is given by

$$\boldsymbol{M}^{-1}(\xi)=\begin{bmatrix} d_{11}^+ & d_{12}^+ & d_{13}^+ \\ d_{21}^+ & d_{22}^+ & d_{23}^+ \\ d_{31}^+ & d_{32}^+ & d_{33}^+ \end{bmatrix}$$

where

$$d_{11}^+=\frac{(w_2w_3+w_1(w_2+w_3))\beta_0^2+w_2(w_1+w_3)\beta_1^2+2w_1w_2\beta_1\beta_2+w_1(w_2+w_3)\beta_2^2+2\beta_0\left(w_2(w_1+w_3)\beta_1\right)+w_1(w_2+w_3)\beta_2}{w_1w_2w_3},$$

$$d_{12}^+=d_{21}^+=\frac{-(w_2+w_3)\beta_0^2+w_2\beta_1^2+2w_2\beta_1\beta_2+(w_2+w_3)\beta_2^2+2\beta_0(w_2\beta_1+(w_2+w_3)\beta_2)}{w_2w_3},$$

$$d_{13}^+=d_{31}^+=\frac{-(w_1+w_3)(\beta_0+\beta_1)^2+2w_1(\beta_0+\beta_1)\beta_2+w_1\beta_2^2}{w_1w_3},$$

$$d_{22}^{+}=\frac{\left(w_2+w_3\right)\beta_0^2+w_2\beta_1^2+2w_2\beta_1\beta_2+\left(w_2+w_3\right)\beta_2^2+2\beta_0\left(w_2\beta_1+\left(w_2+w_3\right)\beta_2\right)}{w_2w_3},$$

$$d_{23}^{+}=d_{32}^{+}=\frac{\left(\beta_0+\beta_1+\beta_2\right)^2}{w_3},$$

$$d_{33}^{+}=\frac{\left(w_1+w_3\right)\left(\beta_0+\beta_1\right)^2+2w_1\left(\beta_0+\beta_1\right)\beta_2+w_1\beta_2^2}{w_1w_3},$$

and

$$\operatorname{Tr}\left(\boldsymbol{M}^{-1}(\xi)\right)=\frac{\left(3w_1w_2+2\left(w_1+w_2\right)w_3\right)\beta_0^2+w_2\left(3w_1+2w_3\right)\beta_1^2)+6w_1w_2\beta_1\beta_2+w_1(3w_1+2w_3)\beta_2^2+\beta_0\left(\left(6w_1w_2+4w_2w_3\right)\beta_1+2w_1\left(3w_2+2w_3\right)\beta_2\right)}{w_1w_2w_3}.$$

Appendix III. Information matrix for Poisson regression model

Case1. When the support points are $\boldsymbol{x}_1^*$, $\boldsymbol{x}_2^*$, and $\boldsymbol{x}_3^*$. The inverse of the information matrix is given by

$$\boldsymbol{M}^{-1}(\xi)=\begin{bmatrix} f_{11}^{+} & f_{12}^{+} & f_{13}^{+} \\ f_{21}^{+} & f_{22}^{+} & f_{23}^{+} \\ f_{31}^{+} & f_{32}^{+} & f_{33}^{+} \end{bmatrix}$$

where $f_{11}^{+}=f_{12}^{+}=f_{21}^{+}=f_{13}^{+}=f_{31}^{+}=f_{23}^{+}=f_{32}^{+}=\frac{e^{-\beta_0}}{w_1}$,

$$f_{22}^{+}=\frac{e^{-\beta_0-\beta_1}\left(w_1+e^{\beta_1}w_2\right)}{w_1w_2},$$

$$f_{33}^{+}=\frac{e^{-\beta_0-\beta_2}\left(w_1+e^{\beta_2}w_3\right)}{w_1w_3}, \text{ and}$$

$$\operatorname{Tr}\left(\boldsymbol{M}^{-1}(\xi)\right)=e^{-\beta_0}\left(\frac{3}{w_1}+\frac{e^{-\beta_1}}{w_2}+\frac{e^{-\beta_2}}{w_3}\right).$$

Case 2. When the support points are $\boldsymbol{x}_1^*$, $\boldsymbol{x}_2^*$, and $\boldsymbol{x}_4^*$. The inverse of the information matrix is given by

$$\boldsymbol{M}^{-1}(\xi)=\begin{bmatrix} g_{11}^{+} & g_{12}^{+} & g_{13}^{+} \\ g_{21}^{+} & g_{22}^{+} & g_{23}^{+} \\ g_{31}^{+} & g_{32}^{+} & g_{33}^{+} \end{bmatrix}$$

where $g_{11}^{+}=g_{12}^{+}=g_{21}^{+}=\frac{e^{-\beta_0}}{w_1}$, $g_{13}^{+}=g_{31}^{+}=0$,

$$g_{22}^{+}=\frac{e^{-\beta_0-\beta_1}\left(w_1+e^{\beta_1}w_2\right)}{w_1w_2},$$

$$g_{33}^{+}=\frac{e^{-\beta_0-\beta_1-\beta_2}\left(w_2+e^{\beta_2}w_3\right)}{w_2w_3},\quad g_{23}^{+}=g_{32}^{+}=\frac{-e^{-\beta_0-\beta_1}}{w_2},$$

and

$$\operatorname{Tr}\left(\boldsymbol{M}^{-1}(\xi)\right)=\frac{e^{-\beta_0-\beta_1-\beta_2}\left(w_1w_2+2e^{\beta_2}(w_1+e^{\beta_1}w_2\right)w_3)}{w_1w_2w_3}.$$

Case 3. When the support points are $\boldsymbol{x}_1^*$, $\boldsymbol{x}_3^*$, and $\boldsymbol{x}_4^*$. The inverse of the information matrix is given by

$$\boldsymbol{M}^{-1}(\xi)=\begin{bmatrix} h_{11}^{+} & h_{12}^{+} & h_{13}^{+} \\ h_{21}^{+} & h_{22}^{+} & h_{23}^{+} \\ h_{31}^{+} & h_{32}^{+} & h_{33}^{+} \end{bmatrix}$$

where $h_{11}^{+}=h_{13}^{+}=h_{31}^{+}=\frac{e^{-\beta_0}}{w_1}$, $g_{12}^{+}=g_{21}^{+}=0$,

$$h_{23}^{+}=h_{32}^{+}=\frac{-e^{-\beta_0-\beta_2}}{w_2},$$

$$h_{22}^{+}=\frac{e^{-\beta_0-\beta_1-\beta_2}\left(w_2+e^{\beta_1}w_3\right)}{w_2w_3},$$

$$h_{33}^{+}=\frac{e^{-\beta_0-\beta_2}\left(w_1+e^{\beta_2}w_2\right)}{w_1w_2},$$

and

$$\operatorname{Tr}\left(\boldsymbol{M}^{-1}(\xi)\right)=e^{-\beta_0}\left(\frac{2}{w_1}+\frac{e^{-\beta_1-\beta_2}\left(w_2+2e^{\beta_1}w_3\right)}{w_2w_3}\right).$$

Case 4. When the support points are $\boldsymbol{x}_2^*$, $\boldsymbol{x}_3^*$, and $\boldsymbol{x}_4^*$. The inverse of the information matrix is given by

$$\boldsymbol{M}^{-1}(\xi)=\begin{bmatrix} m_{11}^{+} & m_{12}^{+} & m_{13}^{+} \\ m_{21}^{+} & m_{22}^{+} & m_{23}^{+} \\ m_{31}^{+} & m_{32}^{+} & m_{33}^{+} \end{bmatrix}$$

where $m_{11}^{+}=\frac{e^{-\beta_0-\beta_1-\beta_2}\left(e^{\beta_2}w_2w_3+w_1\left(w_2+e^{\beta_1}w_3\right)\right)}{w_1w_2w_3}$,

$$m_{12}^{+}=m_{21}^{+}=\frac{-e^{-\beta_0-\beta_1-\beta_2}\left(w_2+e^{\beta_1}w_3\right)}{w_2w_3},$$

$$m_{13}^{+} = m_{31}^{+} = \frac{-e^{-\beta_0-\beta_1-\beta_2}\left(w_1 + e^{\beta_2} w_3\right)}{w_1 w_3},$$

$$m_{22}^{+} = \frac{-e^{-\beta_0-\beta_1-\beta_2}\left(w_2 + e^{\beta_1} w_3\right)}{w_2 w_3},$$

$$m_{23}^{+} = m_{32}^{+} = \frac{-e^{-\beta_0-\beta_1-\beta_2}}{w_3},$$

$$m_{33}^{+} = \frac{e^{-\beta_0-\beta_1-\beta_2}\left(w_1 + e^{\beta_2} w_3\right)}{w_1 w_3},$$

and

$$\mathrm{Tr}\left(\boldsymbol{M}^{-1}(\xi)\right) = \frac{e^{-\beta_0-\beta_1-\beta_2}\left(3w_1 w_2 + 2e^{\beta_1} w_1 w_3 + 2e^{\beta_2} w_2 w_3\right)}{w_1 w_2 w_3}.$$