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On a New Generalized Mixture of Weibull-Rayleigh Distribution

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SUMMARY

In this paper, a novel generalized distribution has been proposed which is obtained through the framework of a θ-weighted mixture family (θ-WM). Two distributions weighted by a parameter θ are combined to create the θ-weighted mixture family, which provides substantial flexibility for modelling a diverse range of intricate phenomena. The new four parameter distribution is referred as θ-Weighted Mixture Weibull-Rayleigh (θ-WMWR) distribution, resulting from the combination of Weibull and Rayleigh distributions. The characterization and numerous statistical properties are derived, including the reliability function, hazard rate function, quantile function, mean residual life, mean inactivity time, r^{th} moment, Rényi entropy, and order statistics. Finally, the applicability and effectiveness of the proposed distribution are demonstrated using real-life datasets, and its performance is compared with existing models such as the Alpha-Power Exponentiated Inverse Rayleigh (APEIR), θ-Weighted Mixture Weibull–Lomax (θ-WMWLx), and Lomax–Rayleigh (LR) distributions to highlight potential of the proposed distribution.

Keywords: Weibull distribution; Rayleigh distribution; Weighted mixture; Survival function; Distribution family.

1. INTRODUCTION

In many applied sciences, modelling and analysing of lifetime data has a significant role. In scientific study, distribution models act as theoretical tools for interpreting complex phenomena, and various lifetime distributions have been utilized to describe this type of data. The standard of the methods employed in a statistical study depends mostly on the assumed distributions or probability model. Consequently, substantial efforts have been devoted in developing broad families of standard probability distributions along with their corresponding statistical methodologies. However, there are still a lot of significant issues where the actual data deviates from any of the conventional or classical probability models. Introducing a new parameter into a distribution model is a perfect instance of a generalization that involves data complexity. The inclusion of new parameters enhances the model's flexibility, allowing it to adapt more effectively to diverse data scenarios.

A significant contribution to this subject was made by Marshall and Olkin [10], whose research established a novel survival function in the following way.

$$\bar{S}(x/\sigma) = \frac{\sigma\bar{F}(x)}{1-\bar{\sigma}\bar{F}(x)}\,;(-\infty < x < \infty; 0 < \sigma < \infty) \quad (1)$$

Where $\bar{\sigma} = 1-\sigma$ and $\bar{F}$ represents the survival function associated with random variable X.

A different approach to create a new distribution family is to use functional transformations. Shama *et al.* [12] proposed the Modified Generalized G family, a flexible distribution built on power-exponential transformations. Their theoretical approach is as follows: let $\bar{G}$ be a survival function of an absolutely continuous distribution with support (0, 1), and H be a decreasing continuous function satisfying $H(x) \in [0,1]$

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and $\lim_{x \to 0} H(x) = 1$. Then, the function defined below is a valid continuous distribution function:

$F(x) = 1 - \overline{G}(x)H(x)$ (2)

The Rayleigh distribution is widely used in various fields such as life testing experiments, reliability studies, applied statistics, and clinical research. Siddiqui [13], Hirano [7], and Howlader and Hussian [8] provide information on the origin and other features of this distribution. Polovko [11] pointed out the significance of this distribution in electrovacuum devices, while Dyer and Whisenand [4] illustrated its significance in communication engineering. Ariyawansa and Templeton [1], Sinha and Howlader [14], and other authors have contributed to this concept.

$$f(x/\sigma) = \sigma x e^{\frac{-\sigma}{2}x^2};x > 0, \sigma > 0$$

Where the cumulative distribution function (CDF) can be expressed as

$$F(x/\sigma) = 1 - e^{\frac{-\sigma}{2}x^2};x > 0, \sigma > 0$$

The Weibull distribution, originally introduced by Weibull [15], is widely applied to model phenomena characterized by monotonic failure rates. However, it is not suitable for data sets exhibiting bathtub-shaped or inverted bathtub-shaped (unimodal) failure rates, which commonly arise in biological, engineering, and reliability studies. To address this limitation, numerous extensions of the Weibull distribution have been proposed in the literature to provide a better fit for such data. For instance, Merovci and Elbatal [10] introduced and analysed the three-parameter Weibull–Rayleigh distribution.

In this study we represent a distribution family called the θ-Weighted mixture distribution using equation (2) which was studied by Carvajal *et al.* [2]. We consider a particular case of θ-weighted mixture family represented by θ-WMWR, which is made up of the Rayleigh and Weibull distributions. We examine the core dependability elements of the θ-WMWR distribution as well as its fundamental statistical characteristics, including survival and hazard function, Renyi entropy, moments, order statistics, and estimation methods. Additionally, we provide two actual datasets for analysing the suggested distribution's performance against three alternative models.

The Rayleigh distribution can be viewed as a particular case of the Weibull distribution when the shape parameter equals 2. By constructing a θ-weighted mixture of the Rayleigh and Weibull distributions, the proposed model can capture a wide range of lifetime behaviors, including decreasing hazard rates (as in the Rayleigh case), increasing hazard rates (as in the Weibull with shape > 1), and even constant or bathtub-shaped hazards depending on the chosen weight and shape parameters. This flexibility allows the model to effectively adapt to different types of reliability data. Moreover, the distribution provides alternative expressions for the density and hazard functions, while the parameter θ influences the skewness and tail behavior of the model.

The paper is outlined as follows. In section 2, we define θ-Weighted Mixture distribution family of distribution. In Section 3, we have introduced our proposed distribution θ-Weighted Weibull-Rayleigh along with a proposition on cumulative distribution function. Reliability study and statistical properties including survival and hazard function, quantile function, moment, order statistics, Renyi entropy and the estimation methods are studied in Section 4. Finally, simulation and applications are demonstrated in section 5 and section 6 respectively.

2. THE θ-WEIGHTED MIXTURE DISTRIBUTION FAMILY (θ WM)

In this section, based on equation (2) the definition of the θ-Weighted Mixture family of distribution (θs-WM) is presented.

Definition 1. Let $F(x \mid a)$ and $G(x \mid b)$ denotes two baseline distribution functions that are absolutely continuous. The survival functions (SF) of $\overline{F}(x \mid a)$ and $\overline{G}(x \mid b)$ are represented by these functions. For $\theta \in [0,1]$, then the associated probability density function (PDF) and cumulative distribution function (CDF) for the θ-WM distribution family:

$$F_\theta(x|a,b) = 1 - \overline{F}(x|a)^{1-\theta}\, \overline{G}(x|b)^{\theta} \;, a,b > 0, x > 0 \quad (3)$$

$$f_\theta\left(x|a,b\right)=\bar{F}\left(x|a\right)^{-\theta}\bar{G}\left(x|b\right)^{\theta}[(1-\theta)f\left(x|a\right)+\theta\bar{F}\left(x|a\right)\bar{G}\left(x|b\right)^{-1}g\left(x|b\right) \quad (4)$$

where f and g denotes the probability density functions of the distributions F and G, respectively.

Remark 1. From the definition 1, we have:

- The parameter θ controls the relative contribution of the survival functions F and G in constructing the resulting distribution.
- If $\theta=0$, then $F_0\left(x|a,b\right)=F(x\,|\,a)$.
- If $\theta=1$, then $F_1\left(x|a,b\right)=G(x\,|\,b)$.

3. θ-WEIGHTED WEIBULL-RAYLEIGH DISTRIBUTION

In this section,we have studied the four parameters θ-Weighted Weibull-Rayleigh (θ-WMWR) distribution.

Let $F(x\,|\,\lambda,k)$ and $G(x\,|\,\sigma)$ be the survival functions of the Weibulland Rayleigh distributions, respectively, that is $F\left(x|\lambda,k\right)=e^{-(\lambda x)^k}$ and $G\left(x|\sigma\right)=e^{-\frac{x^2}{2\sigma^2}}$. Then, according to Definition 1, for $0\le\theta\le1$ we get the cumulative distribution function of the proposed θ-WMWR distribution as

$$F_\theta\left(x|\theta,\lambda,\sigma,k\right)=1-e^{-(1-\theta)(\lambda x)^k}\left(e^{-\frac{x^2}{2\sigma^2}}\right)^{\theta}\quad \lambda,k,\sigma>0,x>0 \quad (5)$$

The corresponding probability density function of the θ-WMWR distribution is given by,

$$f_\theta\left(x|\theta,\lambda,\sigma,k\right)=e^{-(1-\theta)(\lambda x)^k-\frac{x^2\theta}{2\sigma^2}}\left(k\left(1-\theta\right)\lambda^k x^{k-1}+\frac{x\theta}{\sigma^2}\right) \quad (6)$$

Fig. 1 illustrates some of the possible shapes of the pdf and cdf of θ-Weighted Weibull-Rayleigh distribution for selected values of the parameters θ,λ,σ and k respectively.

It is evident from figure 1 that the θ-WMWR distribution shows a variety of shapes in the density function including asymmetric, right skewed and unimodal characteristics.

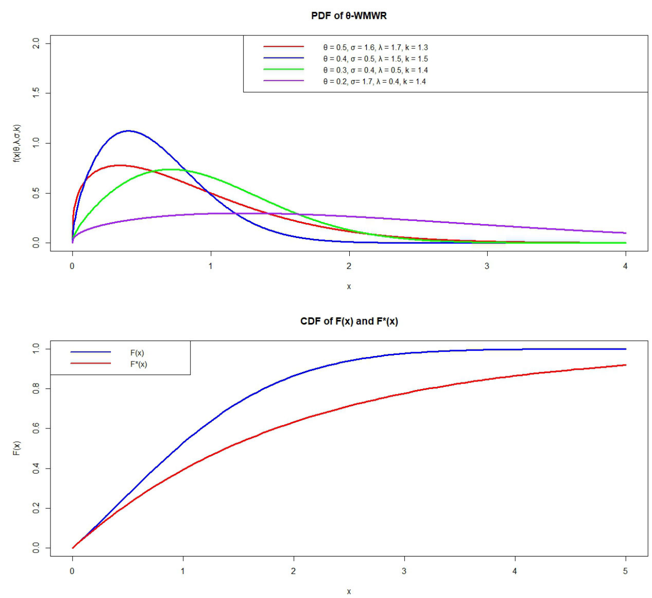


Fig. 1. Plot of pdf and cdf of θ-WMWR distribution

Proposition 1:

Given $F_\theta\left(x\,|\,\theta,\lambda,\sigma,k\right)$ as defined in Equation (5) and $F_\theta^*\left(x\,|\,\theta,\lambda,k\right)=1-[\bar{G}\left(x\,|\,\lambda,k\right)]^{1-\theta}$, it follows that $F_\theta\left(x\,|\,\theta,\lambda,\sigma,k\right)$ exhibits first-order stochastic dominance over $F_\theta^*\left(x\,|\,\theta,\lambda,k\right)$, i.e.,

$$F_\theta\left(x\theta,\lambda,\sigma,k\right)\ge F_\theta^*\left(x\theta,\lambda,k\right),\forall\, x>0.$$

Proof: The above theorem can be proved by subtracting the CDF function as follows,

$$F_\theta\left(x|\theta,\lambda,\sigma,k\right)-F_\theta^*\left(x|\theta,\lambda,k\right)$$

$$=\left(1-e^{-(1-\theta)(\lambda x)^k}\left(e^{-\frac{x^2}{2\sigma^2}}\right)^{\theta}\right)-\left(1-e^{-(1-\theta)(\lambda x)^k}\right)$$

$$=-e^{-(1-\theta)(\lambda x)^k}\left(e^{-\frac{x^2}{2\sigma^2}}\right)^{\theta}+e^{-(1-\theta)(\lambda x)^k}$$

$$=e^{-(1-\theta)(\lambda x)^k}\left(1-\left(e^{-\frac{x^2}{2\sigma^2}}\right)^{\theta}\right)$$

We know that $e^{-(1-\theta)(\lambda x)^k}>0$ for all x, so if $\left(1-\left(e^{-\frac{x^2}{2\sigma^2}}\right)^{\theta}\right)\ge0$ for all $x>0$, stochastic dominance holds. The expression $\left(1-\left(e^{-\frac{x^2}{2\sigma^2}}\right)^{\theta}\right)\geq 0$ is satisfied whenever $\left(e^{-\frac{x^2}{2\sigma^2}}\right)^{\theta}\leq 1$, which is always true for $x\in\mathbb{R},\sigma>0$ and $0\leq\theta\leq 1$. This proves the result.

In figure 1 (right side)we have shown the comparation of two cdf functions and shows the first order stochastic dominance property of $F_{\theta}(x)$ and $F_{\theta}^{*}(x)$.

4. RELIABILITY ELEMENTS AND STATISTICAL PROPERTIES

In this section, we present a comprehensive study of the statistical properties of the θ-WMWR distribution, including survival and hazard functions, mean residual life and inactivity time, quantile function, Rényi entropy, moments, order statistics, estimation techniques, and reliability measures.

4.1 Survival and Hazard functions:

We obtain the survival function, hazard rate, and reverse hazard rate of the suggested θ-WMWR distribution, which are crucial components for describing phenomena in real life. The following is the survival function of the θ-WMWR distribution:

$$R(x)=1-F(x)=e^{-(1-\theta)(\lambda x)^k}\left(e^{-\frac{x^2}{2\sigma^2}}\right)^{\theta}$$

The expressions for the hazard rate and the reverse hazard rate of the model are derived in equations (7) and (8), respectively.

$$h(x)=\frac{f(x)}{R(x)}=\left(k(1-\theta)\lambda^k x^{k-1}+\frac{x\theta}{\sigma^2}\right) \tag{7}$$

$$R.H.R=\frac{f(x)}{F(x)}=\frac{e^{-(1-\theta)(\lambda x)^k-\frac{x^2\theta}{2\sigma^2}}\left(k(1-\theta)\lambda^k x^{k-1}+\frac{x\theta}{\sigma^2}\right)}{1-e^{-(1-\theta)(\lambda x)^k-\frac{x^2\theta}{2\sigma^2}}}$$

$$=\frac{\left(k(1-\theta)\lambda^k x^{k-1}+\frac{x\theta}{\sigma^2}\right)}{e^{(1-\theta)(\lambda x)^k+\frac{x^2\theta}{2\sigma^2}}-1} \tag{8}$$

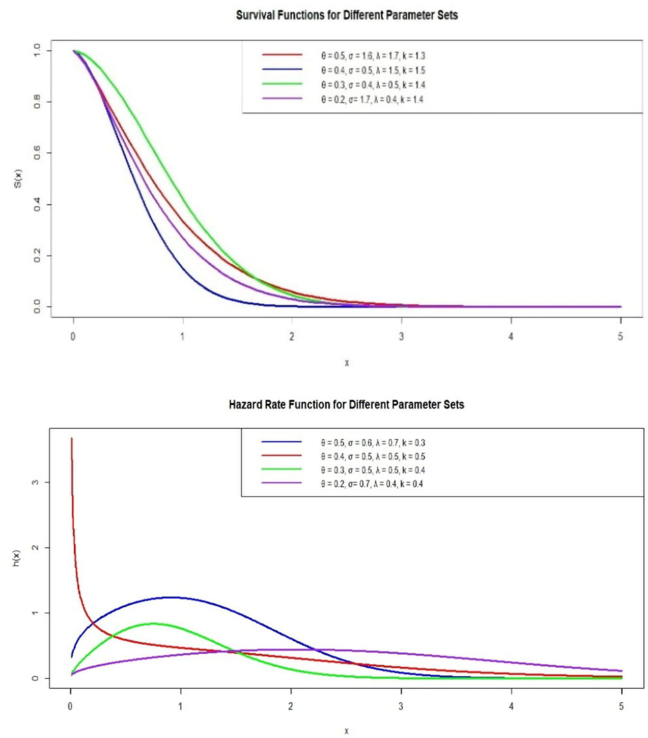


Fig. 2. Plot of Survival function and Hazard function of θ-WMWR distribution

Fig.2 illustrate that the survival function of the model is always a decreasing function for different values of the parameters. Again, on the right side some of the possible shapes of the hazard function of θ-WMWR distribution are shown for selected values of the parameters θ,λ,σ and k respectively.

4.2 Mean residual life:

The mean residual life (MRL) represents expected additional time after a component has survived up to a time τ. The MRL is essential for reliability and survival analysis which describes the process of aging. When the life of a component is represented by the random variable X, the MRL is given by

$$\eta_T(\tau)=\frac{1}{\bar{F}(\tau)}\int_{\tau}^{\infty}\bar{F}(x)\,dx,\tau\geq 0$$

The mean residual life (MRL) function for a lifetime random variable X following the θ-WMWR distribution is expressed as follows

$$\eta_T(\tau \mid \theta,\sigma,\lambda,k) = \frac{1}{e^{-(1-\theta)(\lambda\tau)^k}\left(e^{-\frac{\tau^2}{2\sigma^2}}\right)^{\theta}} \int_{\tau}^{\infty} e^{-(1-\theta)(\lambda x)^k}\left(e^{-\frac{x^2}{2\sigma^2}}\right)^{\theta} dx,\ \tau \geq 0.$$

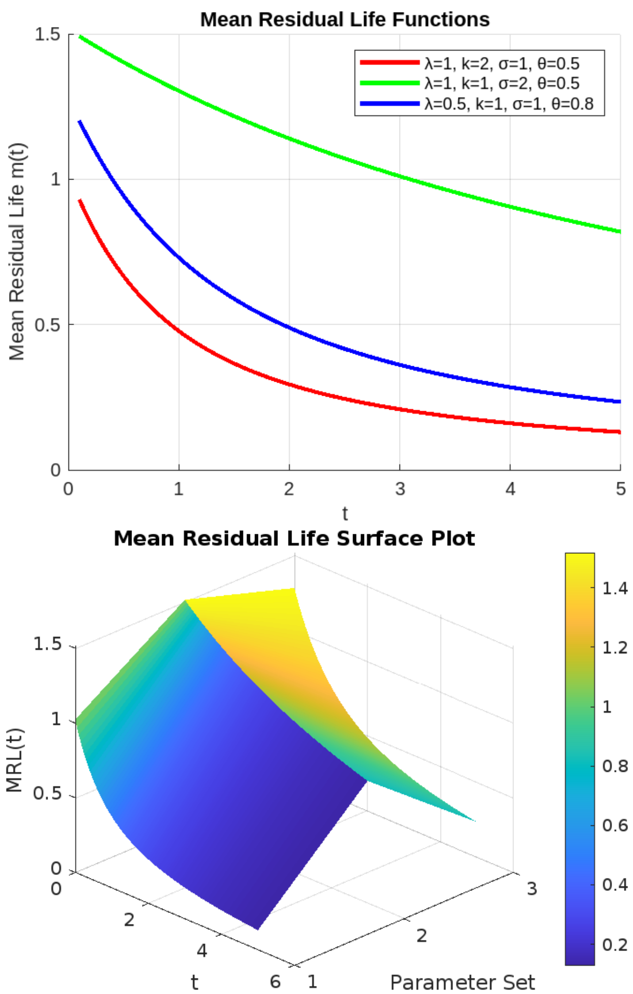


Fig. 3. Plot of Mean Residual life

The mean residual life is illustrated in Fig. 3, with a 3D surface plot displayed on the left and its corresponding 2D plot on the right.

4.3 Mean inactivity time:

The mean inactivity time function serves as an important reliability measure with applications across various domains such as actuarial science, survival analysis, and reliability engineering. The mean inactivity time describes the amount of time since a failure occurred. Assume that X is a random variable with a distribution function F. The mean inactivity time function of X is given by

$$\mu(\tau) = \frac{1}{F(\tau)} \int_{o}^{\tau} F(x)\,dx, \tau > 0$$

The mean inactivity time function of a lifetime random variable X with θ-WMWR distribution is given by

$$\mu(\tau \mid \theta,\sigma,\lambda,k) = \frac{1}{e^{-(1-\theta)(\lambda\tau)^k}\left(e^{-\frac{\tau^2}{2\sigma^2}}\right)^{\theta}} \int_{0}^{\tau} e^{-(1-\theta)(\lambda x)^k}\left(e^{-\frac{x^2}{2\sigma^2}}\right)^{\theta} dx$$

Fig. 4 demonstrates the mean inactivity time of the random variable with different parameter values.

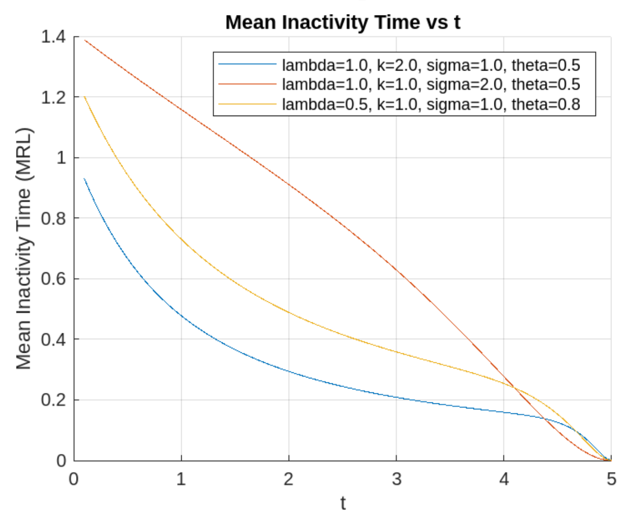


Fig. 4. Plot of Mean inactivity time Vs time

4.4 Quantile function:

In statistical research, the quantile function is extensively used to compute quartiles and generate random numbers for simulation studies. The CDF is inverted to yield the quantile function. To invert the CDF and obtain the quantile function, suppose $F(x) = u$ for x, where $F(x)$ is the given cumulative distribution function (CDF) of θ- WMWR distribution and $u \sim U[0,1]$. Thus $u = 1 - e^{-(1-\theta)(\lambda x)^k}\left(e^{-\frac{x^2}{2\sigma^2}}\right)^{\theta}$.

In this case, the quantile function is obtained as the solution to a non-linear equation.

$$(1-\theta)(\lambda x)^k + \frac{x^2\theta}{2\sigma^2} + \ln(1-u) = 0$$

By assigning $u = 0.5$ in the above expression, we can determine the median (M_d) of the θ-WMWR distribution. Similarly, the first and third quartiles are obtained by setting $u = 0.25$ and $u = 0.75$ respectively. These quantiles can be computed numerically using a root-finding approach, implemented through the uniroot() function in R (version 4.3.3).

4.5 Renyi Entropy:

The uncertain variation of a random variable X is measured by its entropy. The Rényi entropy is given by,

$$I_{\mathbb{R}}(\nu) = \frac{1}{1-\nu}\log\left[I(\nu)\right]$$

Where, $I(\nu) = \int_{\mathbb{R}} f^{\nu}(x)\,dx, \nu \in \mathbb{R}^{+} - \{1\} > 0.$

Let, $\theta \sim \mathrm{WMWR}(\theta, \lambda, \sigma, k)$, the corresponding Rényi entropy is obtained as

$$I_{\mathbb{R}}(\nu) = \frac{1}{1-\nu}\log\left[\int_0^{\infty} e^{-\nu(1-\theta)(\lambda x)^k - \frac{\nu x^2\theta}{2\sigma^2}}\left(k(1-\theta)\lambda^k x^{k-1} + \frac{x\theta}{\sigma^2}\right)^{\nu} dx\right]$$

For computing the Renyi entropy, the numerical integration approach utilizing the integral() function in MATLAB can be applied. Table 1 presents the Renyi entropy values for the θ-WMWR distribution with parameters $\alpha = 2, \sigma = 1, \lambda = 1, k = 2$ and varying values of θ. Fig. 5 shows the Renyi entropy values for different values of θ.

Table 1. Renyi entropy forθ-WMWR

θ	α	λ	k	Renyi Entropy for $\alpha = 2$
0	1	1	2	0.4694604
0.2	1	1	2	0.5218349
0.4	1	1	2	0.5804367
0.6	1	1	2	0.6469298
0.8	1	1	2	0.7237509
1	1	1	2	0.8146775

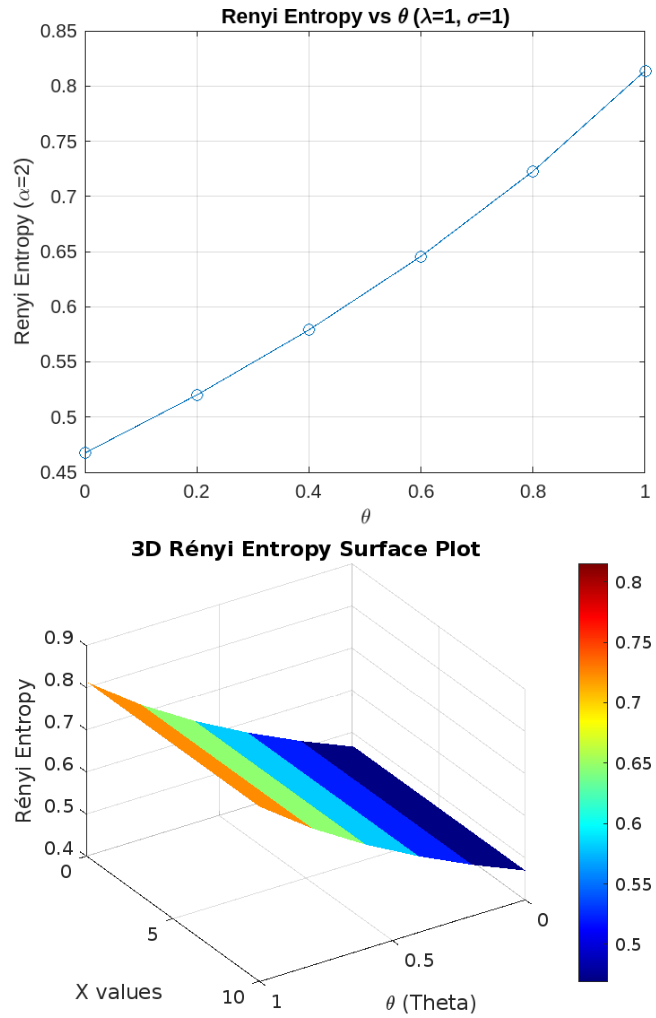


Fig. 5. Plot of Renyi Entropies for different values of θ

4.6 Density Expansion:

This subsection presents an infinite mixture representation of the probability density function associated with the θ-WMWR distribution, which will be utilized in the subsequent analyses.

The exponential series expansion is given by,

$$e^{-x} = \sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!} \tag{9}$$

By using the equation (9) to the pdf of θ-WMWR, we get

$$f_{\theta}(x|\theta, \lambda, \sigma, k) = \sum_{j=0}^{\infty}\sum_{m=0}^{\infty} v_{j,m} w(x; j, m) \tag{10}$$

Where,

$$v_{j,m} = \frac{(-1)^{j+m}}{j!m!}(1-\theta)^j \theta^m \lambda^{kj}$$

$$w(x;j,m) = k(1-\theta)\lambda^k x^{kj+2m+k-1} + \theta\frac{x^{kj+2m+1}}{\sigma^2}$$

4.7 Moments:

If X is continuous non negative random variable with probability density function $f_\theta(x|\theta,\lambda,\sigma,k)$, then the r$^{\text{th}}$ moment of x for the given model is obtained by using the following formula,

$$\mu_r^{'} = E(x^r) = \int_0^\infty x^r f_\theta(x|\theta,\lambda,\sigma,k)\,dx \qquad (11)$$

Substituting equation (10) into equation (11) we get,

$$\mu_r^{'} = \int_0^\infty x^r \sum_{j=0}^\infty\sum_{m=0}^\infty v_{j,m} w(x;j,m)\,dx$$

$$\mu_r^{'} = \sum_{j=0}^\infty\sum_{m=0}^\infty v_{j,m}\int_0^\infty x^r w(x;j,m)\,dx$$

$$\mu_r^{'} = \sum_{j=0}^\infty\sum_{m=0}^\infty v_{j,m}\left[\begin{array}{l} k(1-\theta)\lambda^k\int_0^\infty x^{r+kj+2m+k-1}dx + \\ \theta\frac{1}{\sigma^2}\int_0^\infty x^{r+kj+2m+1}dx \end{array}\right]$$

The integral of x^a over $[0,\infty)$ is:

$$\int_0^\infty x^a dx = \begin{cases} \infty \text{ if } a \le -1 \\ \frac{1}{a+1} \text{ if } a > -1 \end{cases}$$

For the given cases,

$$\int_0^\infty x^{r+kj+2m+k-1}dx = \frac{1}{r+kj+2m+k}$$

And

$$\int_0^\infty x^{r+kj+2m+1}dx = \frac{1}{r+kj+2m+2}$$

Substituting these into the expression of $\mu_r^{'}$,

$$\mu_r^{'} = \sum_{j=0}^\infty\sum_{m=0}^\infty v_{j,m}\left[\frac{k(1-\theta)\lambda^k}{r+kj+2m+k} + \frac{\theta}{\sigma^2(r+kj+2m+2)}\right]$$

Which is the general expression for the r$^{\text{th}}$ moments.

From the above, the mean of X is given by $E(X) = \mu = \mu_1^{'}$ and variance $Var(X) = E(X^2) - [E(X)]^2$.

The r$^{\text{th}}$ central moment is given by,

$$\mu_r = \left[E(X-\mu)^r\right] = \sum_{j=0}^{r}\binom{r}{j}(-1)^j \mu^j \mu_{r-j}^{'} \qquad (12)$$

By using Equation (12), we can obtain the skewness and kurtos is of X as follows

$$S_k = \frac{\mu_3}{\mu_2^{3/2}} \text{ and } K_t = \frac{\mu_4}{\mu_2^2}.$$

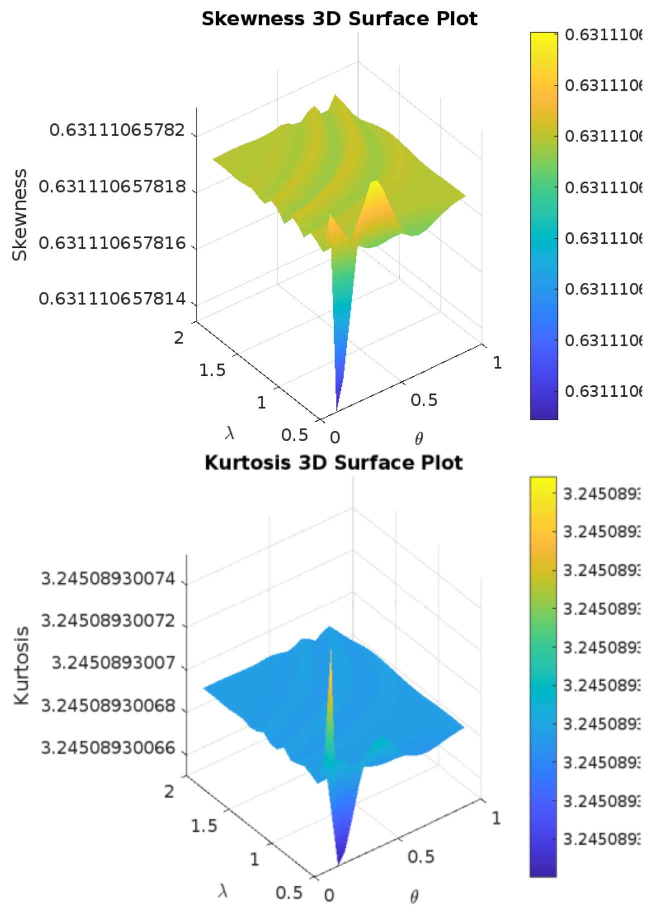


Fig. 6. presents the skewness and kurtosis plot for some specific parameter values.

Fig. 6. Plot of Survival Skewness and Kurtosis of θ-WMWR distribution

4.8 Order statistics:

Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a population having cumulative distribution function (CDF) $F(x)$ and probability density function (PDF)

$f(x)$. Then, the probability density function of the r th order statistic $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is expressed as follows:

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!}\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r} f(x),$$

for $1 \le r \le n$ (13)

Substituting the equations (5) and (6) in equation (13), we get the probability density function rth order statistics of the θ-WMWR distribution as follows

$$= \frac{n!}{(r-1)!(n-r)!}\left[1-e^{-(1-\theta)(\lambda x)^k - \frac{x^2\theta}{2\sigma^2}}\right]^{r-1}\left[e^{-(1-\theta)(\lambda x)^k - \frac{x^2\theta}{2\sigma^2}}\right]^{n-r} \times$$
$$e^{-(1-\theta)(\lambda x)^k - \frac{x^2\theta}{2\sigma^2}}\left(k(1-\theta)\lambda^k x^{k-1} + \frac{x\theta}{\sigma^2}\right)$$

For $r=1$, we get the probability density function of the first order statistics $X_{(1)} = \min\left(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\right)$ as follows

$$f_1(x) = n\left[e^{-(1-\theta)(\lambda x)^k - \frac{x^2\theta}{2\sigma^2}}\right]^{n-1} e^{-(1-\theta)(\lambda x)^k - \frac{x^2\theta}{2\sigma^2}} \times$$
$$\left(k(1-\theta)\lambda^k x^{k-1} + \frac{x\theta}{\sigma^2}\right)$$

Similarly, for $r=n$ we get the probability density function of the first order statistics $X_{(1)} = \min\left(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\right)$ as follows

$$f_n(x) = n\left[1-e^{-(1-\theta)(\lambda x)^k - \frac{x^2\theta}{2\sigma^2}}\right]^{n-1} e^{-(1-\theta)(\lambda x)^k - \frac{x^2\theta}{2\sigma^2}} \times$$
$$\left(k(1-\theta)\lambda^k x^{k-1} + \frac{x\theta}{\sigma^2}\right)$$

For $r = m+1$ and n=2m+1the PDF of median of θ-WMWRdistribution is obtained as follows

$$f_{m+1}(x) = \frac{(2m+1)!}{(m+1)!m!}\left[1-e^{-(1-\theta)(\lambda x)^k - \frac{x^2\theta}{2\sigma^2}}\right]^{m} \times$$
$$\left(e^{-m(1-\theta)(\lambda x)^k + \frac{x^2\theta}{2\sigma^2}}\right) e^{-(1-\theta)(\lambda x)^k - \frac{x^2\theta}{2\sigma^2}}\left(k(1-\theta)\lambda^k x^{k-1} + \frac{x\theta}{\sigma^2}\right)$$

4.9 Maximum Likelihood Estimation:

The unknown parameters of the θ-WMWR distribution are estimated using the maximum likelihood estimation (MLE) method. The corresponding maximum likelihood estimates (MLEs) of the proposed model's parameters are derived as follows:

Let $X_1, X_2, \ldots, X_n$ denote a random sample drawn from the θ-WMWR distribution. Accordingly, the likelihood function and its corresponding log-likelihood function derived from equation (6) are respectively expressed in the following equations.

$$L(x|\theta,\lambda,\sigma,k) = \prod_{i=1}^{n} e^{-(1-\theta)(\lambda x)^k - \frac{x^2\theta}{2\sigma^2}}\left(k(1-\theta)\lambda^k x^{k-1} + \frac{x\theta}{\sigma^2}\right)$$

$$\log L(x|\theta,\lambda,\sigma,k) = \sum_{i=1}^{n}\left[\begin{array}{c} -(1-\theta)(\lambda x_i)^k - \frac{x_i^2\theta}{2\sigma^2} + \\ \log\left(k(1-\theta)\lambda^k x_i^{k-1} + \frac{x_i\theta}{\sigma^2}\right)\end{array}\right]$$

The four corresponding normal equations are obtained by differentiating the Log Likelihood function with respect to the four unknown parameters and setting them to zero as follows:

$$\frac{dlogL(x|\theta,\lambda,\sigma,k)}{d\theta} = \sum_{i=1}^{n}\left[\begin{array}{c}(\lambda x_i)^k - \frac{x_i^2}{2\sigma^2} + \\ \frac{\left(-k\lambda^k x_i^{k-1}\right) + \frac{x_i}{\sigma^2}}{k(1-\theta)\lambda^k x_i^{k-1} + \frac{x_i\theta}{\sigma^2}}\end{array}\right] = 0$$

$$\frac{dlogL(x|\theta,\lambda,\sigma,k)}{d\lambda} = \sum_{i=1}^{n}\left[\begin{array}{c}-(1-\theta)k(\lambda x_i)^{k-1} x_i + \\ \frac{k(1-\theta)k\lambda^{k-1}x_i^{k-1}}{k(1-\theta)\lambda^k x_i^{k-1} + \frac{x_i\theta}{\sigma^2}}\end{array}\right] = 0$$

$$\frac{dlogL(x|\theta,\lambda,\sigma,k)}{d\sigma} = \sum_{i=1}^{n}\left[\frac{\theta x_i^2}{\sigma^3} - \frac{2\theta x_i}{\sigma^3\left(k(1-\theta)\lambda^k x_i^{k-1} + \frac{x_i\theta}{\sigma^2}\right)}\right] = 0$$

$$\frac{dlogL\left(x|\theta,\lambda,\sigma,k\right)}{dk}=\sum_{i=1}^{n}\left[-\left(1-\theta\right)\left(\lambda x_i\right)^k\log\left(\lambda x_i\right)+\left(\frac{1}{k}+\log\lambda+\log x_i\right)\right]=0$$

The simultaneous solutions of the four normal equations provide the maximum likelihood estimates of the unknown parameters $\left(\theta,\lambda,\sigma,k\right)$ of the proposed distribution.

5. SIMULATION STUDY

In this section, we investigate the performance of MLEs of the θ-WMWR distribution parameters by using the Monte Carlo simulation method. In order to analyse Bias and MSE, comprehend how the distribution behaves under various parameter regimes, we employ parametric settings in simulation for the θ-WMWR distribution. We consider a random sample of size $n=50$; 100 and 200 from our density corresponding to particular choices of the parameters as follows $\theta=0.5$, $\lambda=1$, $\sigma=1$, $k=2$. Table 2 provides the Bias and MSE under the method of maximum likelihood estimation. We consider 1, 000 replications for drawing random samples each of size $n=50$; 100 and 200 drawn from our density respectively.

Table 2. Bias and MSE of the parameter estimate using MLE

Sample size		Bias	MSE
50	$\hat{\theta}$	-0.1206873	0.1744855
	$\hat{\sigma}$	-0.0564677	0.0427038
	$\hat{\lambda}$	0.36148374	0.5322641
	$\hat{k}$	9.08867012	0.0397203
100	$\hat{\theta}$	-0.0935269	0.1644723
	$\hat{\sigma}$	-0.0496371	0.0388509
	$\hat{\lambda}$	0.34407310	0.5324249
	$\hat{k}$	9.06456266	0.0539167
200	$\hat{\theta}$	-0.0491143	0.1498646
	$\hat{\sigma}$	-0.0357121	0.0295628
	$\hat{\lambda}$	0.28754536	0.5115695
	$\hat{k}$	3.55318763	0.0149520

The r package can be employed to directly maximize the log-likelihood function, or alternatively, the optim function and Max-BFGS algorithms can be utilized to numerically solve the nonlinear likelihood equations obtained by differentiating the pdf in equation (6). The parameter estimates are derived by equating the score vector to zero and solving it using appropriate statistical software. The results indicate that the maximum likelihood estimation (MLE) method performs effectively, as evidenced by the relatively small values of standard error (SE) and mean squared error (MSE). Moreover, both SE and MSE consistently decrease with increasing sample size. Overall, the results confirm that the MLE approach is suitable for estimating the parameters of the θ-WMWR distribution.

6. APPLICATIONS

In this section, the proposed distribution is fitted to two real-life datasets obtained from Dey *et al.* (2017) and Ghazal and Radwan (2022), respectively. The performance of the proposed model is compared with the Alpha-Power Exponentiated Inverse Rayleigh (APEIR), θ-Weighted Mixture Weibull Lomax (WMWLx), and Lomax Rayleigh (LR) distributions. To evaluate and compare the goodness of fit of the θ-WMWR distribution with these competing models, various statistical measures such as the Kolmogorov–Smirnov (KS) and Anderson–Darling (AD) statistics, along with several information criteria based on the log-likelihood function, have been employed. The discrimination criterion are

Akaike information criterion, AIC = $-2l+2k$

Bayesian information criterion,

BIC = $-2l+\log\left(n\right)k$

Consistent Akaike information criterion,

CAIC = $AIC+k\log\left(n\right)-k$

Where, $n=$ sample size, $k=$ number of parameters to be estimated, $l=$ maximized log likelihood function.

The probability density functions of the mentioned distributions to be compared with the proposed distribution are presented here

APEIR:

$$f\left(x \mid \alpha,\sigma,\theta\right)=\begin{cases}\dfrac{\log\alpha}{\alpha-1}\dfrac{2\sigma\theta}{x^{3}}e^{-\frac{\sigma\theta}{x^{2}}}\alpha^{e^{-\frac{\sigma\theta}{x^{2}}}} & \text{if } \alpha>0,\alpha\neq 1\\ 0, & \text{otherwise}\end{cases}$$

WMWLx:

$$f\left(x \mid \theta,\lambda,k,\alpha,\sigma\right)=e^{\lambda x(\theta-1)-\theta(\sigma x)^{\alpha}}\Big[(1-\theta)\lambda+\theta\alpha\sigma\left(\sigma x\right)^{\alpha-1}\Big];\ \lambda,k,\alpha,\sigma>0, x>0, 0\leq\theta\leq 1$$

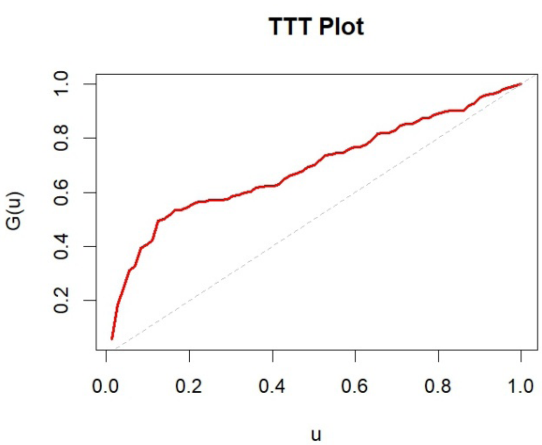
LR: $$f\left(x \mid \theta,\alpha\right)=\frac{2\alpha\theta^{\alpha}x}{\left(\theta+x^{2}\right)^{\alpha+1}};\theta,\alpha>0$$

The necessary numerical computations were performed using the R software package. The maximum likelihood estimates (MLEs) of the parameters, along with the corresponding log-likelihood, AIC, BIC, and CAIC values, are presented below.

APPLICATION I:

The first data sets represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli.

Data set 1: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.



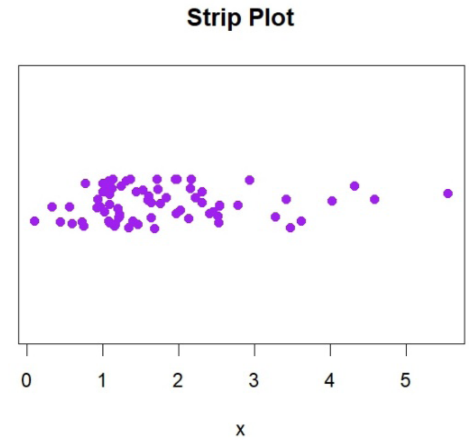
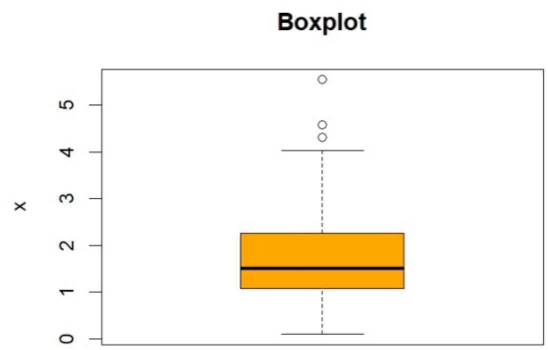


Fig. 7. TTT and Strip plot forthe dataset 1



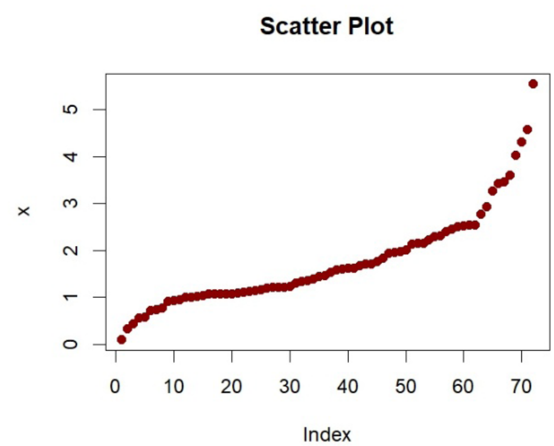


Fig. 8. Scatter plot and Box plot forthe dataset 1

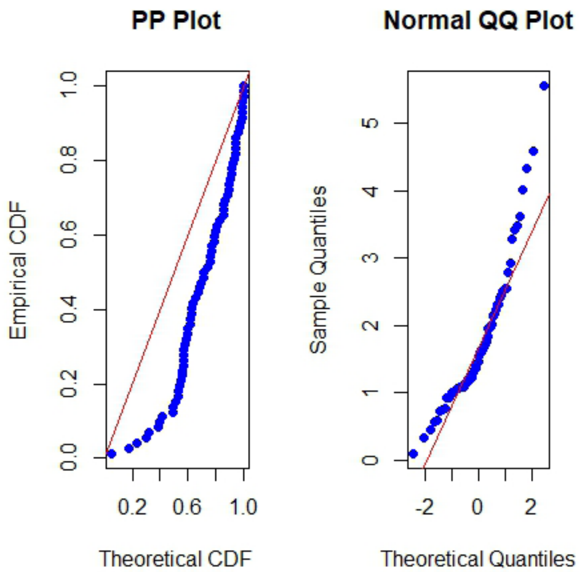


Fig. 9. P-P and Normal Q-Q plot of the θ-WMWR distribution for datasets 1

Fig. 7 presents the TTT and strip plots for dataset 1, while Fig. 8 displays the corresponding scatter and box plots. The scatter, strip, and box plots collectively reveal a predominantly central cluster with a few high-valued outliers. The TTT plot indicates a decreasing failure rate, suggesting that early-life failures dominate, and items that survive the initial period tend to last longer. Fig. 9 illustrates the Normal P–P and Q–Q plots, and Fig. 10 shows the fitted density curve for dataset 1.

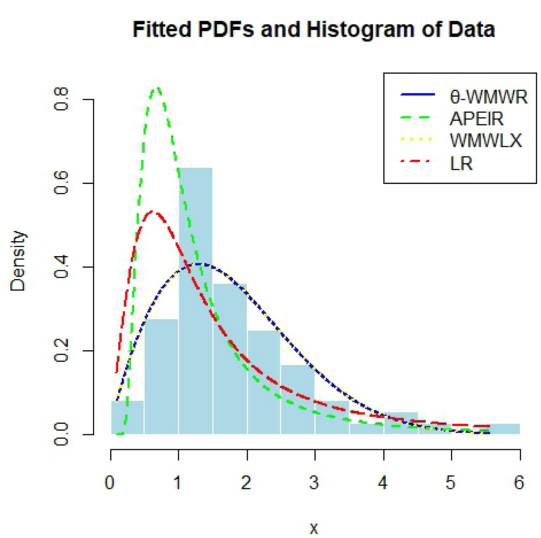


Fig. 10. Fitted densities of the θ-WMWR distribution for data sets 1

Table 3. MLE for dataset 1

Models	$\hat{\theta}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{k}$	$\hat{\alpha}$
θ-WMWR	0.9835938	1.6585938	-0.1812500	2.0492187	-
APEIR	0.0602951	0.0001976	-	-	0.0004551
WMWLx	1.06298993	0.32854794	0.09707059	1.2505995	1.5409035
LR	14.837319	-	-	-	4.539985

Table 4. Goodness of fit for dataset 1

Models	-2log L	AIC	BIC	CAIC	A	D
θ-WMWR	**191.5808**	**199.5808**	**208.6874**	**220.6874**	**12.23528**	**0.9806972**
APEIR	264.7473	270.7443	277.5773	286.5773	12.45343	0.9908017
WMWLx	191.5798	201.5798	212.9631	227.9631	23.23537	0.9966945
LR	216.8197	220.8197	225.3731	231.3731	22.62088	0.9807628

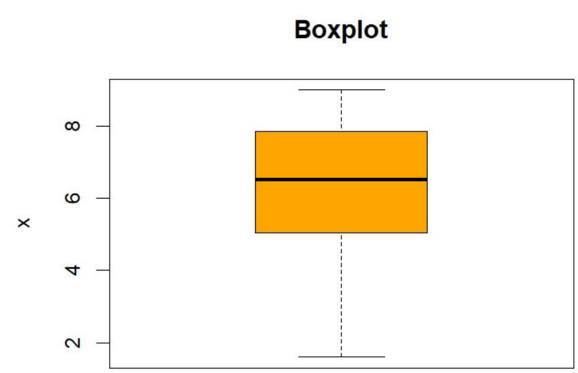
Table 3 and Table 4 illustrate the MLE estimated values of the parameters and the goodness of fit measures for the datasets 1 respectively. The results in Table 4 indicate that, in comparison with its competitors, the θ-WMWR distribution consistently provides superior performance by achieving the minimum AIC, BIC, and CAIC values.

Application II:

The second dataset is related to test records of the time-to-failure data for $n = 40$ suits of turbochargers.

Data set 2:1.6, 2, 2.6, 3, 3.5, 3.9, 4.5, 4.6, 4.8, 5, 5.1, 5.3, 5.4, 5.6, 5.8, 6, 6 ,6.1, 6.3, 6.5, 6.5, 6.7 ,7, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8 ,8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.

The θ-WMWR distribution is fitted to the above data sets, and its goodness of fit has been compared with those of its special cases.



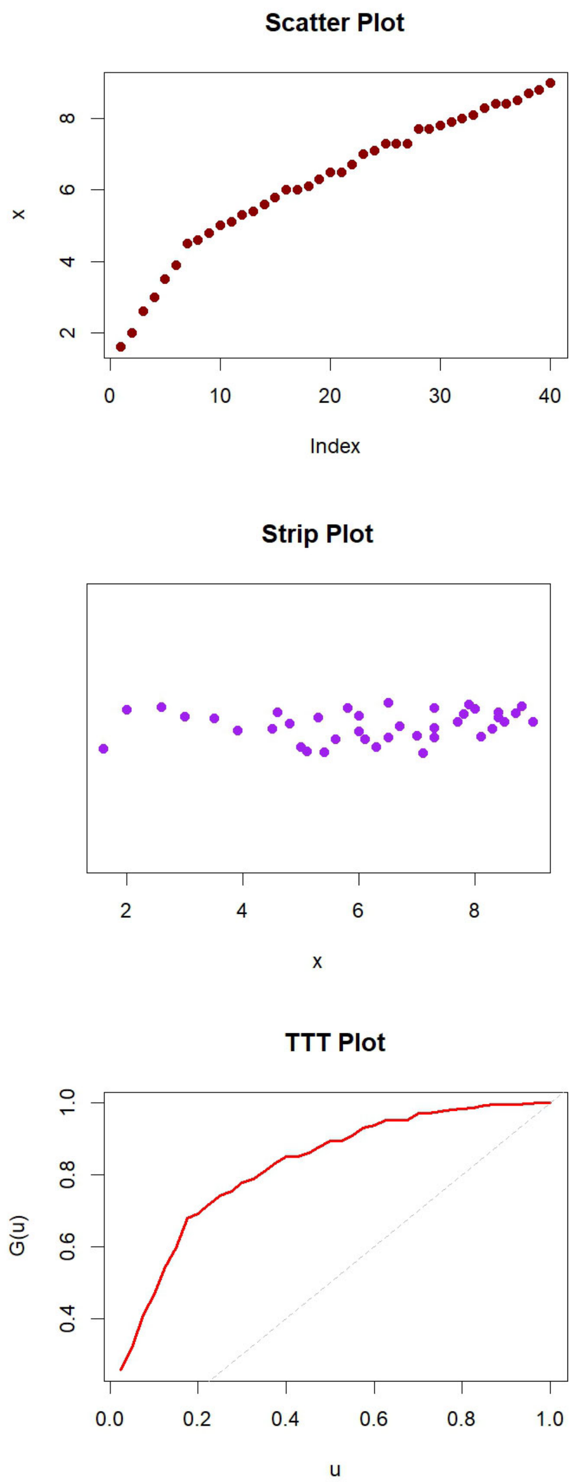


Fig. 11. TTT, Strip plot, Scatter plot and box plot for the dataset 2

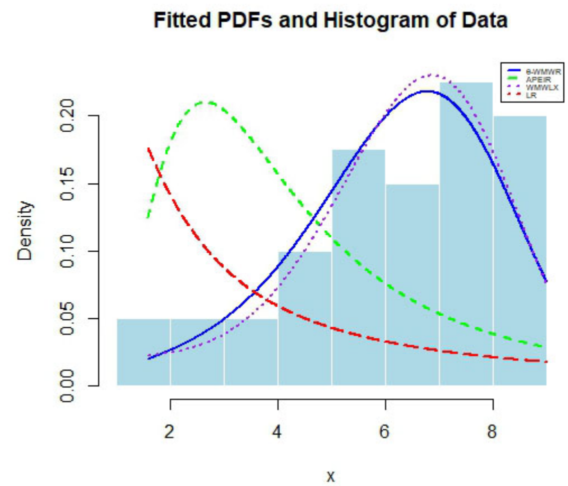


Fig. 12. Fitted densities of the θ-WMWR distribution for data sets 2

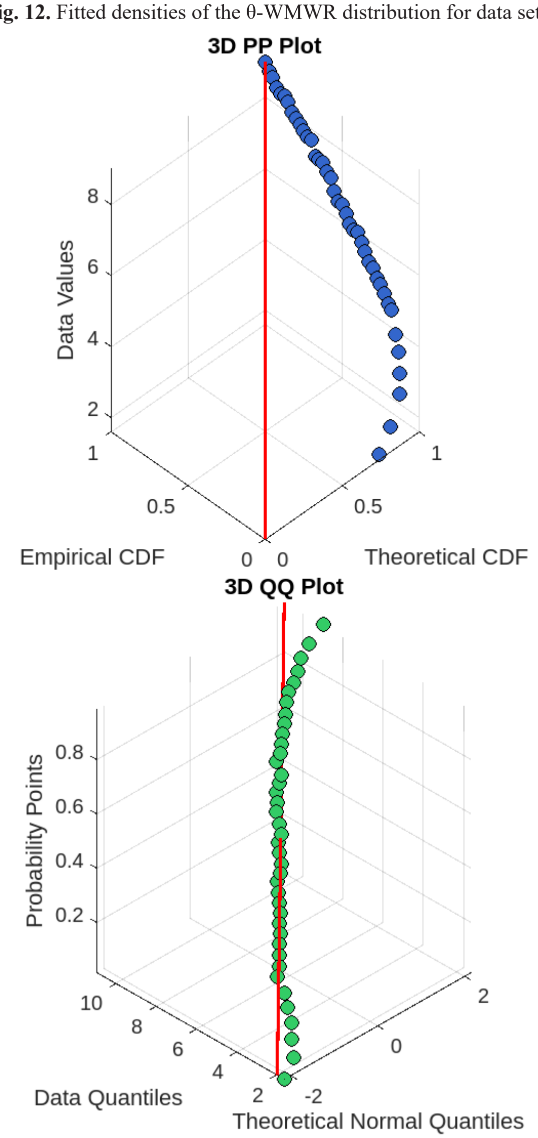


Fig. 13. 3D P-P and Normal Q-Q plot of the θ-WMWR distribution for datasets 2

Fig. 11 displays various plot representing the dataset such as box-plot, strip plot, scatter plot along with the TTT plot for the dataset 2. The dataset shows a central concentration around mid-range values with moderate variation and no significant outliers. The TTT plot reveals a decreasing failure rate, implying early failures followed by a more stable reliability phase. Fig. 12 shows the fitted density curve with the dataset 2. Fig. 13 shows the P-P and Q-Q plot of the θ-WMWR distribution for datasets 2.

Table 5. MLE for dataset 2

Models	-2log L	AIC	BIC	CAIC	A	D
θ-WMWR	**161.359**	**169.359**	**176.1145**	**188.1145**	**34.46664**	**0.9224133**
APEIR	215.6487	221.6487	226.7154	235.7154	56.73275	0.9714859
WMWLx	162.3606	172.3606	180.805	195.805	34.59323	0.9238578
LR	271.5318	275.5318	278.9096	284.9096	85.12117	0.9821338

Table 6. Goodness of fit for dataset 2

Models	$\hat{\theta}$	$\hat{\sigma}$	$\hat{\lambda}$	$\hat{k}$	$\hat{\alpha}$
θ-WMWR	1.1058594	1.5433594	0.5042969	1.2218750	-
APEIR	36.568612	0.167276	-	-	4.097295
WMWLx	1.0801092	0.6495925	1.0152397	1.2612676	0.1146955
LR	1.859375	-	-	-	-0.137500

Table 5 and Table 6 illustrate the MLE estimated values of the parameters and the goodness of fit measures for the datasets 2 respectively. The results obtained in table 6 highlighted the superior performance of the θ-WMWR.

7. CONCLUSIONS

In this manuscript, the family of θ-weighted mixture of distributions is introduced, providing a versatile framework for modeling the joint behavior of random variables. A novel model known as the θ-Weighted Mixture Weibull-Rayleigh (θ-WMWR) distribution has been presented encompassing the pdf of θ-weighted mixture distribution family. The model is designed to address the increasing demand for flexible tools that can handle complex and diverse types of data. The θ-WMWR distribution is best suited for non-negative, right-skewed continuous data where flexibility in hazard behaviour (increasing, decreasing, or non-monotonic) is required. It generalizes Weibull and Rayleigh, so if those models are too restrictive, θ-WMWR can often provide a better fit. Some of the statistical properties such as moments,survival function, and hazard rate function have been studied for our proposed distribution. The mean residual life, mean inactivity time, Renyi entropies, skewness, and kurtosis have been shown graphically using MATLAB. The Maximum likelihood estimation of the unknown parameters involved in the model has been studied and numerical method was used to estimate the parameters. A simulation study has confirmed the estimators are efficient and reliable under different conditions, supporting the model's use in standard statistical inference. The various discrimination criteria have been used to verify the superior performance of our suggested model, as the optimal model is the one that provides the lowest values of those criteria. Finally, two real data sets have been considered in order to make comparison between special cases of θ-WMWR in terms of fitting. Further extension of the proposed model is also possible and will be studied in future work.

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