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Construction of Constant Block-Sum Designs through Confounded Factorials

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SUMMARY

This article focus on concept of constant block-sum designs through confounded factorial. Confounded factorial designs are shown to provide a rich class of constant block-sum designs. The approach given by Khattree (2022) provides a direct and straightforward proof of the necessary condition for existence of constant block-sum designs. Based on this the method of construction of constant block-sum designs through confounded factorial given by Gupta (2022) is also mentioned. Some designs given by Gupta (2022) are estimated at different values, along with that some more designs based on various effects confounded in different replications are presented to show existence and non-existence of constant block-sum designs in different replications.

Keywords: Balanced Incomplete Block Design; Treatment contrast; Confounding; Factorial experiments.

1. INTRODUCTION

In bio-medical clinical trials and other related research areas, minimizing the use of animals is a priority. In such experiments, it becomes necessary to reuse the same set of animals in several biomedical experiments. But the problem arises while reusing the same set of animals. The same set of animals when reused causes partial damaged due to the application of different treatments in different times and thus contribute some carryover effects in further experimentation. So to overcome that an approach to reduce the effect of partial damage so that it can be used for further experimentation. Concept of constant block-sum has been demonstrated by Khattree (2018).

One may want to ensure that for all the experimental units, the cumulative doses that are subjected to over time, if the treatments were doses of a drug, are constant.

Blocks	Treatments				Sum
B1	10	3	13	8	**34**
B2	5	16	2	11	**34**
B3	4	9	7	14	**34**
B4	16	6	12	1	**34**

After an adequate washout period, these experimental units may then be deemed as equally affected and thus contributing nearly equal carry-over effects in subsequent clinical trial.

Assuming that the treatment levels are quantitative, then the first set of an experiment should be designed ensuring that the sum of treatments over time within a single experimental unit remains constant for all units. Earlier, in case of constant block-sum, constant block-sum BIBD does not exist (Khattree 2018a) while constant block-sum PBIBD not only exist but can also be easily constructed. Khattree (2018b, 2019) showed non-existence of balanced incomplete block designs

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and discussed some methods for obtaining constant block sum PBIB designs.

Recently, factorial experiments are gaining importance as it allow researchers to investigate the effects of multiple factors simultaneously in a single experiment. Factorial designs are used in experimental situations where one way heterogeneity is present in experimental material. It enables the study of interaction effects between factors and also offers flexibility by allowing researchers to explore various combinations of factor levels to better understand their effects on the response variable. There are many real-world situations involve multiple factors acting simultaneously in a complex scenario, for that purpose factorial designs provide accurate results that are more representative of it. For example Dey (1986), in an agricultural study researchers are investigating the effects of soil type (clay vs. loam) and irrigation method (drip vs. sprinkler) on crop yield. By using a factorial design, the researchers can easily estimate the agricultural conditions where crops are grown in different soil types and irrigated using different methods, resulting in findings that are more representative of actual farming practices.

When the number of factors are very large, it is difficult to maintain the homogeneity within single block. Confounding is done in situation where one or more (generally higher order) interactions are mixed or entangled with block comparisons due to which blocks sizes are reduced by making one or more treatment contrast identical with the block contrast.

Bansal and Garg (2022) derived some conditions for existence of partially balanced constant block-sum designs and gave further combinatorial methods of their construction. A general approach to determine whether or not a given design can be transformed into a constant block-sum design and its construction if it exists has been developed in Khattree (2022). Based on that, the purpose here is to study construction of constant block-sum designs using factorial designs. Confounded factorial designs are shown to provide a rich class of constant block-sum designs and this approach also provides a direct and straightforward proof of the necessary condition for existence of constant block-sum designs given by Khattree (2022).

2. PRELIMINARIES AND DEFINITIONS

Consider an equi-replicate confounded block design with parameters v,b,r,k where, v = number of treatments, b = number of blocks, r = number of replications, k = number of treatments in a single block. $\boldsymbol{\tau}=(\tau_1,\tau_2,\cdots,\tau_v)'$ denotes the $v\times1$ vectors of treatment parameters, and $\boldsymbol{\beta}=(\beta_1,\beta_2,\cdots,\beta_b)'$ denotes $b\times1$ vectors of block parameters respectively.

Let $\boldsymbol{h'\tau}$ denote a treatment contrast that is partially or completely confounded in the design, $\boldsymbol{h'}1_v=0$, where 1_v denotes $v\times1$ vector of 1's and $\boldsymbol{s'\tau}$ denotes a treatment contrast that is estimated with full efficiency in the design which is not confounded in any of the replications of the design, $\boldsymbol{s'}1_v=0$. The factorial effects that are estimated with full efficiency are referred as the completely unconfounded effects.

To motivate the method of construction, replace the i^{th} treatment in the confounded design by the i^{th} element of *h* and *s*. In other words, the treatments in the design are replaced by the corresponding coefficients of the confounded and unconfounded contrasts. This can be illustrated with the help of the following example.

Design1 (Gupta, 2022):

Consider the 2^3 partially confounded design of having parameters $v=8, b=4, r=2, k=4$. The designed is obtained by confounding the three-factor interaction *ABC* in first replication and the two-factor interaction *BC* in the other replication.

Table 1

ABC confounded		*BC* confounded	
Block 1	**Block 2**	**Block 3**	**Block 4**
000	001	000	001
101	010	011	010
110	100	100	101
011	111	111	110

Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, $\boldsymbol{u}_{12}$, $\boldsymbol{u}_{13}$, $\boldsymbol{u}_{23}$, and $\boldsymbol{u}_{123}$ be the contrast coefficient vectors for *A, B, C* main effects and *AB, AC, BC, ABC* interaction effects respectively.

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_{12}' \\ u_{13}' \\ u_{23}' \\ u_{123}' \end{bmatrix} = \begin{bmatrix} -1+1-1-1+1+1-1+1 \\ -1-1+1-1+1-1+1+1 \\ -1-1-1+1-1+1+1+1 \\ +1-1-1+1+1-1-1+1 \\ +1-1+1-1-1+1-1+1 \\ +1+1-1-1-1-1+1+1 \\ -1+1+1+1-1-1-1+1 \end{bmatrix}$$

Also, the vector of treatment parameters can be written as,

$\boldsymbol{\tau}' = (\tau_{000}\tau_{001}\tau_{010}\tau_{011}\tau_{100}\tau_{101}\tau_{110}\tau_{111})$

where τ_x denoting the effect of the treatment combination x. Now replace the treatment combinations in each block by the corresponding *ABC* contrast coefficients and obtain the design displayed in Table 2. The block sums are given in the last row of the table.

Table 2. Replace treatment combinations by the corresponding ABC contrast coefficients

	ABCconfounded		**BC confounded**	
	Block 1	**Block 2**	**Block 3**	**Block 4**
	-1	+1	-1	+1
	-1	+1	-1	+1
	-1	+1	+1	-1
	-1	+1	+1	-1
Block sums	**-4**	**+4**	**0**	**0**

Similarly, Tables 3 gives the design obtained by replacing the treatment combinations in each block of the design by respectively the *BC*.

Table 3. Replace treatment combinations by the corresponding BC contrast coefficients

	ABC confounded		**BC confounded**	
	Block 1	**Block 2**	**Block 3**	**Block 4**
	+1	-1	+1	-1
	-1	-1	+1	-1
	-1	+1	+1	-1
	+1	+1	+1	-1
Block sums	**0**	**0**	**+4**	**-4**

Similarly, Table 4 give the designs obtained by replacing the treatment combinations in each block of the design by respectively *AB* contrast coefficients.

Table 4. Replace treatment combinations by the corresponding AB contrast coefficients

	ABC confounded		**BC confounded**	
	Block 1	**Block 2**	**Block 3**	**Block 4**
	+1	+1	+1	+1
	-1	-1	-1	-1
	+1	-1	-1	-1
	-1	+1	+1	+1
Block sums	**0**	**0**	**0**	**0**

Block sums in Table 4 are constant, being equal to zero, corresponding to the *AB* interaction estimated with full efficiency in the design and similar corresponding to the other four unconfounded effects *A, B, C* and *AC* respectively. However, block sums are not constant for the designs of Tables 2 and 3 corresponding to the partially confounded interactions *ABC* and *BC* respectively. So a property of constant block-sum design can be defined as follows:

Property1 (Gupta 2022), When block contents of a partially confounded design are replaced by corresponding coefficients of a treatment contrast then the property of constant block sum being equal to zero holds for all contrasts that are estimated with full efficiency. Furthermore, this property does not hold for the treatment contrasts that are or completely confounded in the design.

3. METHOD OF CONSTRUCTION (GUPTA, 2022)

Now discussing construction of constant block-sum designs through confounding factorial:

Step 1- Take q as the number of treatment contrasts that are estimated with full efficiency in a factorial design that are denoted by $\boldsymbol{U'\tau}$ where,

$$\boldsymbol{U'\tau} = \begin{bmatrix} \boldsymbol{u}_1' \\ \boldsymbol{u}_2' \\ . \\ . \\ \boldsymbol{u}_q' \end{bmatrix} \boldsymbol{\tau}$$

and $\boldsymbol{u}'_i = (\boldsymbol{u}_{i1}\ \ \boldsymbol{u}_{i2} \ldots\ \boldsymbol{u}_{iv})$, with $\boldsymbol{u}_i' 1_v = 0$, $i = 1, 2, \ldots, q$.

Step 2- Estimation of $\boldsymbol{t}_u'$:

Firstly consider $\boldsymbol{C'} = (c_1 \;\; c_2 \; \dots \; c_q)$, where and c_i's are some constants chosen such that all the elements of t_u are different from each other, and

$$\boldsymbol{t}'_{\boldsymbol{u}} = \left(\sum_{i=1}^{q} c_i u_{i1} \sum_{i=1}^{q} c_i u_{i2} \;\; \dots \;\; \sum_{i=1}^{q} c_i u_{iq} \right)$$

Step 3- Estimation of $\boldsymbol{t}_{\boldsymbol{u}}^{*\prime}$:

Now consider θ_u, a linear function of the q contrasts given by,

$$\theta_u = \boldsymbol{C}'\boldsymbol{U}'\boldsymbol{\tau} = \sum_{i=1}^{q} (c_i \boldsymbol{u}'_i)\boldsymbol{\tau} = \boldsymbol{t}'_u \boldsymbol{\tau}$$

But $\boldsymbol{t}_{\boldsymbol{u}}$ being a contrast coefficient vector, $\sum_{i=1}^{v} t_{ui}$ = 0, which means that not all the t_{ui}'s are greater than zero.

Property 2 (Khattree, 2022), constant block-sum still holds if a constant value, say c_0, is added to all the elements of $\boldsymbol{t}_{\boldsymbol{u}}$ to make it greater than zero.

$\boldsymbol{t}_{\boldsymbol{u}}^{*\prime} = (\, t_{u1} + c_0 \;\; t_{u2} + c_0 \;\; \dots \;\; t_{uv} + c_0 \,)$

Step 4- Estimation of θ_u:

Now, replace the elements of $\boldsymbol{t}_{\boldsymbol{u}}$ by $\boldsymbol{t}_{\boldsymbol{u}}^{*}$ for making it greater than zero,

$$\theta_u = \boldsymbol{t}_{\boldsymbol{u}}^{*\prime} \boldsymbol{\tau}$$

Being a linear function of treatment contrasts that are estimated with full efficiency, the treatment contrast θ_u is also estimated with full efficiency.

Step 5- Based on the property 1 that constant block-sum holds when block contents of the design are replaced by corresponding treatment effects in the linear function θ_u, replace the treatment combinations in the design by the corresponding elements of $\boldsymbol{t}_{\boldsymbol{u}}^{*\prime}$ to attain constant block sum design.

4. ILLUSTRATION

Considering the 2^3 partially confounded design.

Design1 (Gupta, 2022):

Considering ABC confounded in first replication and BC confounded in second replication.

Here five completely unconfounded contrasts *A, B, C, AB* and *AC*, i.e. $q=5$, given by

$\boldsymbol{U} = (\boldsymbol{u}_1 \boldsymbol{u}_2 \boldsymbol{u}_3 \boldsymbol{u}_{12} \boldsymbol{u}_{13})$

Let $\boldsymbol{T}$ denote the vector of treatment combinations arranged in the lexicographic order,

$\boldsymbol{T'}$ = (000 001 010 011 100 101 110 111)

Taking $\boldsymbol{C'}$ = (0.44 -0.10 -0.08 0.18 -0.20)

Estimation of $\boldsymbol{t}'_{\boldsymbol{u}}$:

Firstly, by taking all the unconfounded treatment combination contrast coefficient vectors i.e. $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, $\boldsymbol{u}_{12}$ and $\boldsymbol{u}_{13}$ as follows,

$$\begin{bmatrix} \boldsymbol{u}'_1 \\ \boldsymbol{u}'_2 \\ \boldsymbol{u}'_3 \\ \boldsymbol{u}'_{12} \\ \boldsymbol{u}'_{13} \end{bmatrix} = \begin{bmatrix} -1 & +1 & -1 & -1 & +1 & +1 & -1 & +1 \\ -1 & -1 & +1 & -1 & +1 & -1 & +1 & +1 \\ -1 & -1 & -1 & +1 & -1 & +1 & +1 & +1 \\ +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} +0.44 \\ -0.10 \\ -0.08 \\ +0.18 \\ -0.20 \end{bmatrix}$$

Now, calculating

$$t_{000} = \sum_{i=1}^{q} c_i u_{i1} = c_1 u_{11} + c_2 u_{21} + c_3 u_{31} + c_{12} u_{(12)1} + c_{13} u_{(13)1}$$

t_{000} = (-0.44+0.10+0.08+0.18-0.20) = -0.28

Similarly estimating all the elements of t'_u:

t_{001}	=	(	+0.44	+	0.1	+	0.08	-	0.18	+	0.20	)	=	+0.64
t_{010}	=	(	-0.44	-	0.1	+	0.08	-	0.18	-	0.20	)	=	-0.84
t_{011}	=	(	-0.44	+	0.1	-	0.08	+	0.18	+	0.20	)	=	-0.04
t_{100}	=	(	+0.44	-	0.1	+	0.08	+	0.18	+	0.20	)	=	+0.80
t_{101}	=	(	+0.44	+	0.1	-	0.08	-	0.18	-	0.20	)	=	+0.08
t_{110}	=	(	-0.44	-	0.1	-	0.08	-	0.18	+	0.20	)	=	-0.60
t_{111}	=	(	+0.44	-	0.1	-	0.08	+	0.18	-	0.20	)	=	+0.24

$\boldsymbol{t}'_{\boldsymbol{u}}$ = (-0.28 +0.64 -0.84 -0.04 +0.80 +0.08 -0.60 +0.24)

Since $\boldsymbol{t}_{\boldsymbol{u}}$ being the contrast coefficient vector, so

$t_{000} + t_{001} + \ldots + t_{111} =$ (-0.28+0.64-0.84-0.04+0.80+0.08-0.60+0.24) = 0

Estimation of $\boldsymbol{t}_{\boldsymbol{u}}^{*}$:

For calculating $\boldsymbol{t}_{\boldsymbol{u}}^{*\prime}$, add a constant c_0 in such a way that all the elements of $\boldsymbol{t}_{\boldsymbol{u}}^{\prime}$ becomes greater than zero.

Here, assuming c_0 = 1.20,

$\boldsymbol{t}_{\boldsymbol{u}}^{*\prime} = (\; t_{000} + c_0 \;\; t_{001} + c_0 \;\ldots\; t_{111} + c_0 \;)$

So $\boldsymbol{t}_{\boldsymbol{u}}^{*\prime}$ = (0.92 1.16 0.36 0.60 1.84 1.28 1.28 2.00 1.44)

Replacing the i^{th} element of *T* in Table 1 by the i^{th} element of $\boldsymbol{t}_{\boldsymbol{u}}^{*\prime}$, (i= 001, 010, … 111)

Table 5

	ABC confounded		BC confounded	
	Block 1	**Block 2**	**Block 3**	**Block 4**
	0.92	1.16	0.92	1.16
	1.28	0.36	0.60	0.36
	2.00	1.84	1.84	1.28
	0.60	1.44	1.44	2.00
Block sums	**4.80**	**4.80**	**4.80**	**4.80**

This yields a design with a constant block-sum of $4c_0 = 4.8$.

Property 3 (Gupta, 2022)**,** if constant block-sum design exists then the sum of elements of each block after replacing the i^{th} element of $\boldsymbol{T}$ by the i^{th} element of $\boldsymbol{t}_{\boldsymbol{u}}^{*\prime}$ is equal to the product of c_0 (i.e. 1.2) and the number of treatments combinations present in the each block (i.e. 4).

Property 4 (Gupta, 2022)**,** the sum of all the values of $\boldsymbol{t}_{\boldsymbol{u}}^{*\prime}$ is equal to product of c_0 and total number of treatments combinations in an individual replication.

5. EMPIRICAL STUDY

Design 2: Consider a $\mathbf{2^4}$ partially confounded design presented in Table 6, having parameters *v* = 16, *b* = 8, *r* = 2, *k* = 4, where

Effects Confounded	Independent		Generalised
First Replication	*ABC*	*BCD*	*AD*
Second Replication	*ABD*	*ACD*	*BC*
Third Replication	*AB*	*CD*	*ABCD*

Table 6

Replication 1				Replication 2				Replication 3			
B1	**B2**	**B3**	**B4**	**B5**	**B6**	**B7**	**B8**	**B9**	**B10**	**B11**	**B12**
0000	0001	0011	0010	0000	0010	0011	0001	0000	0001	0100	0101
0110	0111	0101	0100	0111	0101	0100	0110	0011	0010	0111	0110
1011	1010	1000	1001	1001	1011	1010	1000	1100	1101	1000	1001
1101	1100	1110	1111	1110	1100	1101	1111	1111	1110	1011	1010

Contrast coefficient vectors for main effects and interaction effects be (Illustration 1):

Table 7. Replace treatment combinations by the corresponding ABC contrast coefficients

Replication 1				Replication 2				Replication 3			
B1	**B2**	**B3**	**B4**	**B5**	**B6**	**B7**	**B8**	**B9**	**B10**	**B11**	**B12**
-1	+1	+1	-1	-1	+1	-1	+1	-1	+1	-1	+1
-1	+1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1
-1	+1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1
-1	+1	+1	-1	-1	+1	-1	+1	-1	+1	-1	+1
-4	**+4**	**+4**	**-4**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Table 8. Replace treatment combinations by the corresponding AC contrast coefficients

Replication 1				Replication 2				Replication 3			
B1	**B2**	**B3**	**B4**	**B5**	**B6**	**B7**	**B8**	**B9**	**B10**	**B11**	**B12**
+1	-1	+1	-1	-1	+1	-1	+1	+1	+1	-1	-1
-1	+1	-1	+1	+1	-1	+1	-1	-1	-1	+1	+1
+1	+1	-1	-1	+1	-1	+1	-1	-1	-1	+1	+1
-1	-1	+1	+1	-1	+1	-1	+1	+1	+1	-1	-1
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

The completely unconfounded *q = 6* contrast coefficient vectors are given by,

$$\boldsymbol{U} = \left(\boldsymbol{u}_1 \boldsymbol{u}_2 \boldsymbol{u}_3 \boldsymbol{u}_4 \;\; \boldsymbol{u}_{13} \boldsymbol{u}_{24} \;\right)$$

$$\boldsymbol{C'} = \left(\boldsymbol{c}_1 \boldsymbol{c}_2 \boldsymbol{c}_3 \boldsymbol{c}_4 \;\; \boldsymbol{c}_{13} \boldsymbol{c}_{24} \;\right)$$

$$\boldsymbol{C'} = \left(+0.25 + 0.35 \; -0.20 \; -0.40 + 0.50 \; -0.30 \right)$$

Estimation of $\boldsymbol{t}_{\boldsymbol{u}}^{\prime}$:

$$\boldsymbol{t}_{\boldsymbol{u}}^{\prime} = \left(\sum_{i=1}^{q} c_i u_{i1}^{\prime} \sum_{i=1}^{q} c_i u'_{i2} \;\; \ldots \;\; \sum_{i=1}^{q} c_i u'_{iq} \right)$$

So, $\boldsymbol{t}_{\boldsymbol{u}}^{\prime}$ = (+0.2 -0.3 +1.5 1 -1.2 0.3 +0.1 +1.6 0.0 -0.5 +0.1 -0.4 -1.4 +0.1 -1.3 +0.2)

As $\boldsymbol{t}_{\boldsymbol{u}}$ being the contrast coefficient vector, so

Illustration 1

$\boldsymbol{u}_1'$	=	(	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1	)
$\boldsymbol{u}_2'$	=	(	-1	-1	+1	+1	-1	-1	+1	+1	-1	-1	+1	+1	-1	-1	+1	+1	)
$\boldsymbol{u}_{12}'$	=	(	+1	-1	-1	+1	+1	-1	-1	+1	+1	-1	-1	+1	+1	-1	-1	+1	)
$\boldsymbol{u}_3'$	=	(	-1	-1	-1	-1	+1	+1	+1	+1	-1	-1	-1	-1	+1	+1	+1	+1	)
$\boldsymbol{u}_{13}'$	=	(	+1	-1	+1	-1	-1	+1	-1	+1	+1	-1	+1	-1	-1	+1	-1	+1	)
$\boldsymbol{u}_{23}'$	=	(	+1	+1	-1	-1	-1	-1	+1	+1	+1	+1	-1	-1	-1	-1	+1	+1	)
$\boldsymbol{u}_{123}'$	=	(	-1	+1	+1	-1	+1	-1	-1	+1	-1	+1	+1	-1	+1	-1	-1	+1	)
$\boldsymbol{u}_4'$	=	(	-1	-1	-1	-1	-1	-1	-1	-1	+1	+1	+1	+1	+1	+1	+1	+1	)
$\boldsymbol{u}_{14}'$	=	(	+1	-1	+1	-1	+1	-1	+1	-1	-1	+1	-1	+1	-1	+1	-1	+1	)
$\boldsymbol{u}_{24}'$	=	(	+1	+1	-1	-1	+1	+1	-1	-1	-1	-1	+1	+1	-1	-1	+1	+1	)
$\boldsymbol{u}_{124}'$	=	(	-1	+1	+1	-1	-1	+1	+1	-1	+1	-1	-1	+1	+1	-1	-1	+1	)
$\boldsymbol{u}_{34}'$	=	(	+1	+1	+1	+1	-1	-1	-1	-1	-1	-1	-1	-1	+1	+1	+1	+1	)
$\boldsymbol{u}_{134}'$	=	(	-1	+1	-1	+1	+1	-1	+1	-1	+1	-1	+1	-1	-1	+1	-1	+1	)
$\boldsymbol{u}_{234}'$	=	(	-1	-1	+1	+1	+1	+1	-1	-1	+1	+1	-1	-1	-1	-1	+1	+1	)
$\boldsymbol{u}_{1234}'$	=	(	+1	-1	-1	+1	-1	+1	+1	-1	-1	+1	+1	-1	+1	-1	-1	+1	)

$t_{0000} + t_{0001} + \ldots + t_{1111} = (+0.2 - 0.3 + 1.5 \ldots +0.2) = 0$

Estimation of $\boldsymbol{t}_{\boldsymbol{u}}^{*\prime}$: Adding a constant c_0 in such a way that all the elements of $\boldsymbol{t}_{\boldsymbol{u}}^{'}$ becomes greater than zero.

Here, assuming c_0 = 2.0,

$\boldsymbol{t}_{\boldsymbol{u}}^{*\prime} = (\ t_{0000}^{'} + c_0 t_{0001}^{'} + c_0 \quad \ldots \ t_{1111}^{'} + c_0\ )$

$\boldsymbol{t}_{\boldsymbol{u}}^{*\prime}$ = (2.2 1.7 3.5 3.0 0.8 2.3 2.1 3.6 2.0 1.5 2.1 1.6 0.6 2.1 0.7 2.2)

The treatment combination vector given by,

$$\boldsymbol{\tau}' = (\tau_{0000}\tau_{0001}\tau_{0010}\tau_{0011}\tau_{0100}\tau_{0101}\tau_{0110}\tau_{0111}\tau_{1000}\tau_{1001} \\ \tau_{1010}\tau_{1011}\tau_{1100}\tau_{1101}\tau_{1110}\tau_{1111})$$

Replace the i^{th} element of $\boldsymbol{T}$ in Table 6 by the i^{th} element of $t_u^{*'}$ to get block sums,

Table 9

Replication 1				Replication 2				Replication 3			
B1	**B2**	**B3**	**B4**	**B5**	**B6**	**B7**	**B8**	**B9**	**B10**	**B11**	**B12**
2.2	1.7	3.0	3.5	2.2	1.7	0.8	2.3	2.2	3.5	3.0	1.7
2.1	3.6	2.3	0.8	3.0	3.5	3.6	2.1	3.6	2.3	0.8	2.1
1.6	2.1	2.0	1.5	0.6	2.1	2.0	1.5	1.5	1.6	2.1	2.0
2.1	0.6	0.7	2.2	2.2	0.7	1.6	2.1	0.7	0.6	2.1	2.2
8.0	**8.0**	**8.0**	**8.0**	**8.0**	**8.0**	**8.0**	**8.0**	**8.0**	**8.0**	**8.0**	**8.0**

This yields a design with a constant block-sum of $4c_0$ = 8.0.

Property 5, if all the main effects and interaction effects are estimated with half or greater than half efficiency then the design will always show the constant block-sum for all replications as shown in Design 1 and 2.

To check existence or non-existence of constant block sum designs when same effect confounded in different replications.

Design 3: Consider a 2^5 partially confounded design presented in Table 10, having parameters $v = 32$, $b = 16$, $r = 2$, $k = 4$ where

Effects Confounded	Independent			Generalised
First Replication	*ABC*	*BCD*	*CDE*	*AD, BE, ACE, ABDE*
Second Replication	*ABD*	*ACD*	*BDE*	*AE, BC, CDE, ABCE*

Table 10

B1	B2	B3	B4	B5	B6	B7	B8
00000	00001	00011	00010	00110	00111	00101	00100
01101	01100	01110	01111	01011	01010	01000	01001
10110	10111	10101	10100	10000	10001	10011	10010
11011	11010	11000	11001	11101	11100	11110	11111
B9	B10	B11	B12	B13	B14	B15	B16
00000	00001	00100	00101	00111	00110	00011	00010
01110	01111	01010	01011	01001	01000	01101	01100
10011	10010	10111	10110	10100	10101	10000	10001
11101	11100	11001	11000	11010	11011	11110	11111

Contrast coefficient vectors for main effects and interaction effects be (Illustration 2):

Table 11 shows design obtained by replacing the treatment combinations in each block of the design by respectively for *CDE*.

Table 11. Replace treatment combinations by the corresponding CDE contrast coefficients

B1	B2	B3	B4	B5	B6	B7	B8
-1	+1	-1	+1	-1	+1	-1	+1
-1	+1	-1	+1	-1	+1	-1	+1
-1	+1	-1	+1	-1	+1	-1	+1
-1	+1	-1	+1	-1	+1	-1	+1
-4	+4	-4	+4	-4	+4	-4	+4
B9	B10	B11	B12	B13	B14	B15	B16
-1	+1	+1	-1	+1	-1	-1	+1
-1	+1	+1	-1	+1	-1	-1	+1
-1	+1	+1	-1	+1	-1	-1	+1
-1	+1	+1	-1	+1	-1	-1	+1
-4	+4	+4	-4	+4	-4	-4	+4

The completely unconfounded $q = 18$ be the contrast coefficient vector and

$\boldsymbol{C}'$ = (+0.25 +0.35 -0.20 -0.35 +0.30 +0.40 +0.30 +0.15 -0.4 -0.35 -0.25 +0.25 +0.15 -0.25 -0.3 +0.25 +0.35 -0.4)

Estimation of $\boldsymbol{t}_u'$: (using MS-Excel) (Illustration 3)

Estimation of $\boldsymbol{t}_u^{*'}$ (Illustration 4):

$\boldsymbol{t}_u^{*'}$ = (3.05 2.25 1.75 1.95 1.65 3.45 3.15 5.15 2.25 2.25 3.35 3.55 2.05 1.85 2.35 3.15 3.15 3.35 4.85 4.85 5.35 2.95 1.85 4.85 4.75 1.75 2.85 5.25 0.15 2.15 1.45 3.25)

Illustration 2

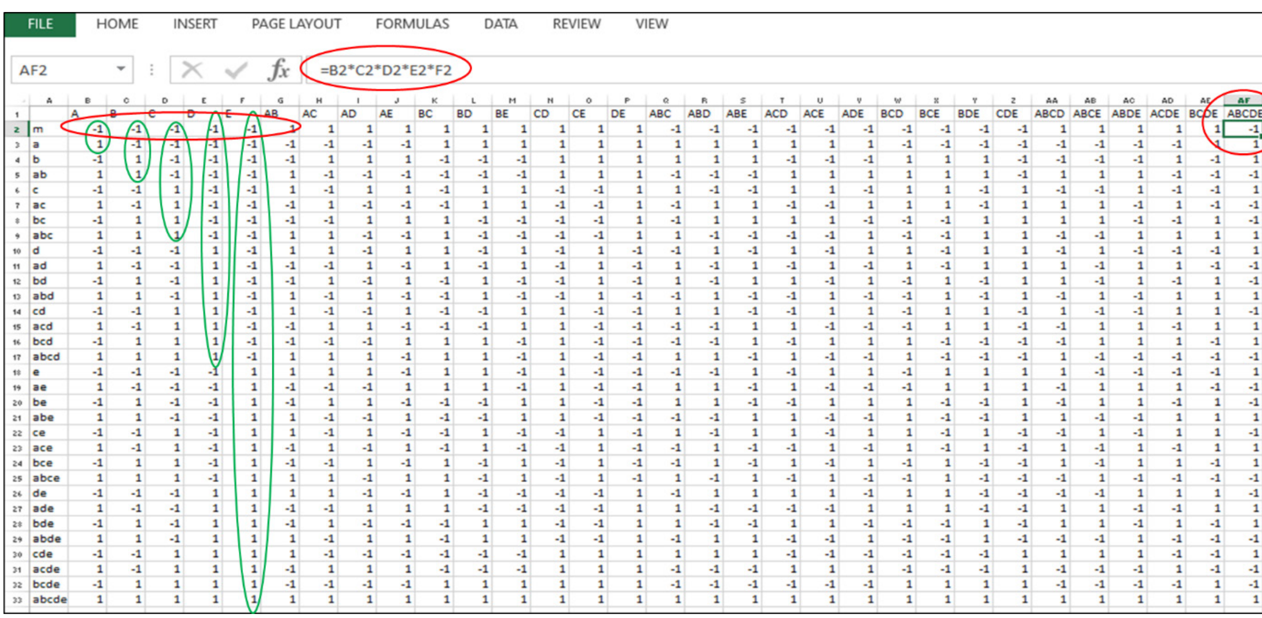


Illustration 3

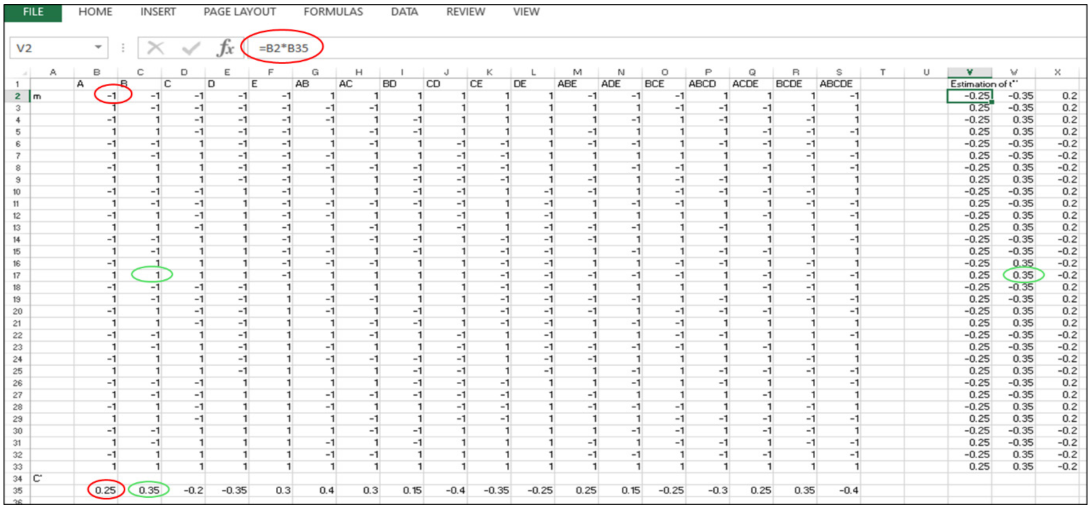
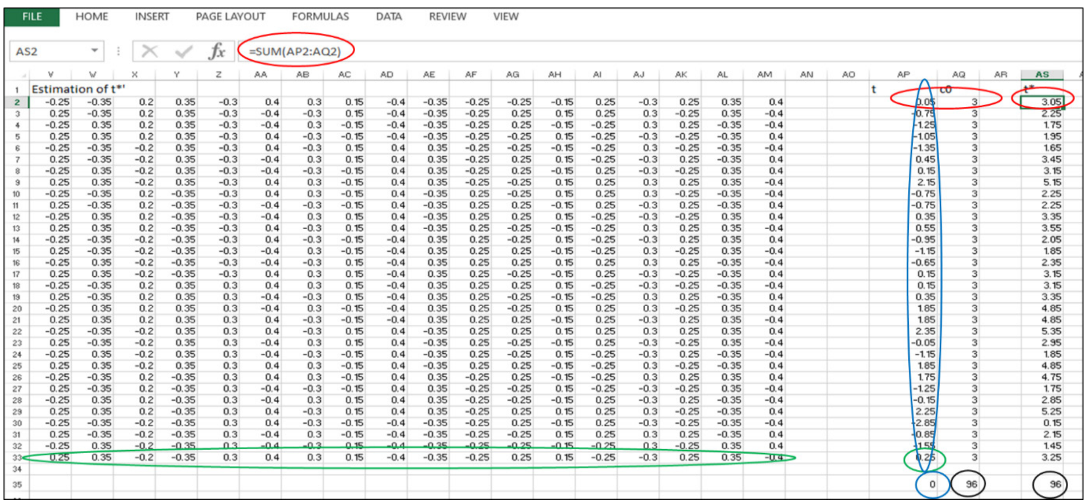


Illustration 4



Replace the i^{th} element of $\boldsymbol{T}$ in Table 10 by the i^{th} element of $t_u^{*'}$ to get block sums,

Table 12

B1	B2	B3	B4	B5	B6	B7	B8
3.05	2.25	1.95	1.75	3.15	5.15	3.45	1.65
1.85	2.05	2.35	3.15	3.55	3.35	2.25	2.25
1.85	4.85	2.95	5.35	3.15	3.35	4.85	4.85
5.25	2.85	4.75	1.75	2.15	0.15	1.45	3.25
12.00	**12.00**	**12.00**	**12.00**	**12.00**	**12.00**	**12.00**	**12.00**
B9	B10	B11	B12	B13	B14	B15	B16
3.05	3.15	5.15	2.25	1.95	3.45	1.65	1.75
2.05	2.25	2.25	3.15	1.85	3.55	3.35	2.05
4.85	2.95	5.35	4.85	3.15	1.85	4.85	3.35
2.15	5.25	2.85	0.15	1.45	4.75	1.75	3.25
12.10	**13.60**	**15.60**	**10.40**	**08.40**	**13.60**	**11.60**	**10.40**

This yields a design with a constant block-sum of $4c_0 = 12$ in first replication but in the second replication the constant block-sum is not attained. As the interaction effect *CDE* is confounded in both the replications or is not estimable in any of the replications causes failure to yield constant block sum design in both the replications replication.

Property 6, addition to Property 4 that for any replication, sum of all the values of $t_u^{*'}$ in an individual replication is equal to product c_0 and total number of treatment combinations whether the constant block-sum exist for the individual replication or not.

So necessary and sufficient condition for constant block sum design to exist is that no effects should have zero efficiency (confounded in all replications) for constant block sum to exist in all replications.

Some more designs where any one of the effect is not estimable in any of the replications.

Design 4:

Effects Confounded	Independent		Generalised	Constant Block-Sum
First Replication	*ABC*	*BCD*	*AD*	Exist
Second Replication	*ABC*	*ACD*	*BD*	Do not exit

Design 5:

Effects Confounded	Independent		Generalised	Constant Block-Sum
First Replication	*ABC*	*BCD*	*AD*	Exist
Second Replication	*ABC*	*ACD*	*BD*	Do not exit
Third Replication	*ABC*	*ABD*	*CD*	Do not exit

Property 7, if same independent effect is confounded in all the replications and no other confounded effect is repeated in any of the replications then constant block sum will exist only in any one the replication as shown in Designs 3, 4 and 5.

Design 6:

Effects Confounded	Independent		Generalised	Constant Block-Sum
First Replication	*AB*	*CD*	*ABCD*	Exist
Second Replication	*ABC*	*BCD*	*AD*	Exist
Third Replication	*ABC*	*ACD*	*BD*	Do not exit

Property 8, if effects confounded in one replication is estimable in all other replications then the constant block sum for that replication will always exist as shown in Design 6.

Design 7:

Effects Confounded	Independent		Generalised	Constant Block-Sum
First Replication	*ABC*	*BCD*	*AD*	Exist
Second Replication	*ABC*	*ACD*	*BD*	Do not exit
Third Replication	*BCD*	*ACD*	*AB*	Do not exit

From Design 7, for 2^4 factorial, if all the independent effects confounded in all replication have same efficiency and then the constant block sum exists for any one the replication only.

Design 8:

Effects Confounded	Independent		Generalised	Constant Block-Sum
First Replication	*ABC*	*BCD*	*AD*	Exist
Second Replication	*ABC*	*ACD*	*BD*	Do not exit
Third Replication	*BCD*	*ACD*	*AB*	Do not exit
Fourth Replication	*ABD*	*ACD*	*BC*	Exist

Design 9:

Effects Confounded	Independent		Generalised	Constant Block-Sum
First Replication	*ABC*	*BCD*	*AD*	Exist
Second Replication	*ABC*	*ACD*	*BD*	Do not exit
Third Replication	*ABD*	*ACD*	*AB*	Exist

An approach to show existence of constant block sum-design in 2^4 factorial experiment when same effect is confounded in two replications.

Design 10:

Effects Confounded	Independent		Generalised	Constant Block-Sum
First Replication	ABC	BCD	AD	Exist
Second Replication	ABC	ACD	BD	Exist
Third Replication	ABD	BCD	AC	Exist

Here, independent effect *ABC* is confounded in replication 1 and 2 so in replication 3 by taking independent effect confounded other than *ABC* in replication 1 and effect which is not confounded in replication 2, we can get the constant block-sum design for all the three replications.

Design 11 (Gupta, 2022)**:** (for different levels)

3^2 partially confounded factorial design with parameters $v=9, b=6, r=2$, $k=3$, where *AB* component in one replication and AB^2 component in the other replication.

Table 13

Replication 1			Replication 2		
Block 1	**Block 2**	**Block 3**	**Block 4**	**Block 5**	**Block 6**
00	01	02	00	02	01
12	10	11	11	10	12
21	22	20	22	21	20

$\boldsymbol{T}' = (00\ 01\ 02\ 10\ 11\ 12\ 20\ 21\ 22)$

The completely unconfounded $q=4$ contrast coefficient vectors are given by,

$$\boldsymbol{U} = \left(u_{1l}u_{1q}u_{2l}u_{2q}\right)$$

$$\boldsymbol{C'} = \left(+0.78\ -0.85\ +0.56\ -0.77\right)$$

Estimation: (Illustration 5)

$$\boldsymbol{t}_u' = (-2.96+0.37-1.4-0.09+3.24+1.47-1.84+1.49-0.28)$$

$$\boldsymbol{t}_u^{*\prime} = (0.04\ 3.37\ 1.6\ 2.91\ 6.24\ 4.47\ 1.16\ 4.49\ 2.72)$$

Replacing the i^{th} element of $\boldsymbol{T}$ in Table 13 by the i^{th} element of $\boldsymbol{t}_u^{*\prime}$,

Table 14

	Replication 1			Replication 2		
	Block 1	**Block 2**	**Block 3**	**Block 4**	**Block 5**	**Block 6**
	0.04	2.91	1.6	0.04	1.6	3.37
	4.47	3.37	6.24	6.24	2.91	4.47
	4.49	2.72	1.16	2.72	4.49	1.16
Block sums	**9.00**	**9.00**	**9.00**	**9.00**	**9.00**	**9.00**

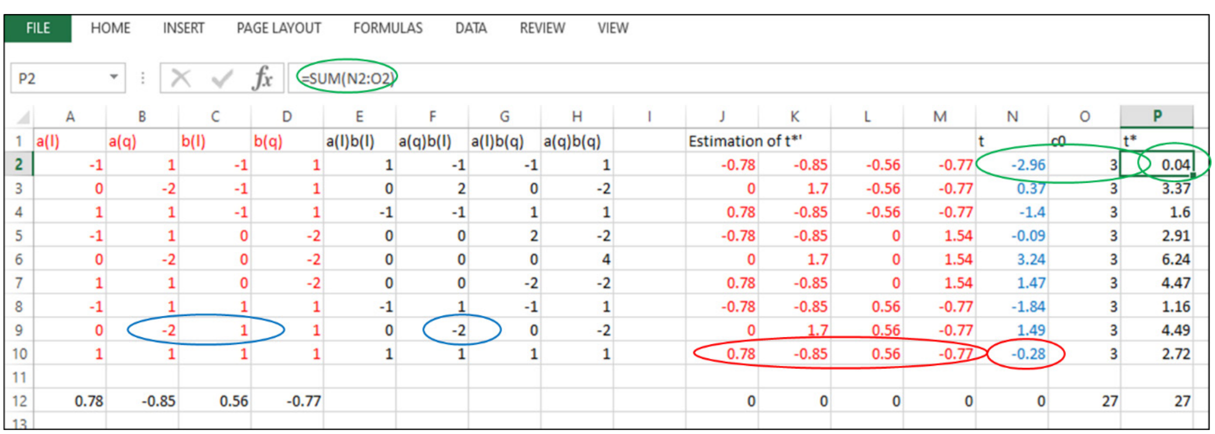
This yields a design with a constant block-sum of $3\,c_0 = 9.00$.

From this, we can conclude that constant block sum designs exists for different factors and at different levels.

6. CONCLUSION

Constant block-sum designs have more importance in dose-response studies in case of animal experiments where a particular animal is used to repeated measurements over a time period w.r.t. doses under investigation. If we wish to do an experiment repeatedly, the set of experimental units used as homogeneous and thus can reuse them in later subsequent experiments. Constant block-sum designs through confounded factorial not only exist but can also be easily constructed by simply arranging the confounded effects in such a way that effects which are confounded in one replication should not be repeated in the other replications. The constant block-sum designs are derived by searching for a treatment levels vector $\boldsymbol{t}_u^{*\prime}$ through trial and error investigation. Different ways for existence and non-existence of constant block sum designs through confounded factorials are discussed here but no such method to find the effects confounded in different replications and also no systematic method to find $\boldsymbol{t}_u^{*\prime}$ is present in the literature. The effects confounded should be based on the choice of the experimenter i.e. interaction effects which are of least interested are confounded and a general approach to find efficient block designs through confounded factorials as well as a systematics method to find the $\boldsymbol{t}_u^{*\prime}$ is required from practical point of view.

Illustration 5



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P2 =SUM(N2:O2)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	a(l)	a(q)	b(l)	b(q)	a(l)b(l)	a(q)b(l)	a(l)b(q)	a(q)b(q)		Estimation of t*'				t	c0	t*
2	-1	1	-1	1	1	-1	-1	1		-0.78	-0.85	-0.56	-0.77	-2.96	3	0.04
3	0	-2	-1	1	0	2	0	-2		0	1.7	-0.56	-0.77	0.37	3	3.37
4	1	1	-1	1	-1	-1	1	1		0.78	-0.85	-0.56	-0.77	-1.4	3	1.6
5	-1	1	0	-2	0	0	2	-2		-0.78	-0.85	0	1.54	-0.09	3	2.91
6	0	-2	0	-2	0	0	0	4		0	1.7	0	1.54	3.24	3	6.24
7	1	1	0	-2	0	0	-2	-2		0.78	-0.85	0	1.54	1.47	3	4.47
8	-1	1	1	1	-1	1	-1	1		-0.78	-0.85	0.56	-0.77	-1.84	3	1.16
9	0	-2	1	1	0	-2	0	-2		0	1.7	0.56	-0.77	1.49	3	4.49
10	1	1	1	1	1	1	1	1		0.78	-0.85	0.56	-0.77	-0.28	3	2.72
11																
12	0.78	-0.85	0.56	-0.77						0	0	0	0	0	27	27
13																

REFERENCES

Bansal, N. and Garg, D.K. (2022). Construction and existence of constant block sum PBIB designs. *Communications in Statistics - Theory and Methods*, **51**(7), 2231-2241.

Dey, A. (1986). *Theory of Block Designs*. 1st Edition, John Wiley and Sons, New York.

Gupta, S. (2022). Constant block-sum designs through confounded factorial. *Statistics and Applications,* **20**(2), 93-101.

Khattree, R. (2018a). A note on the nonexistence of the constant block-sum balanced incomplete block designs. *Communications in Statistics - Theory and Methods*, **48**(20), 5165-5168.

Khattree, R. (2018b). The parshvanathyantram yields a constant-sum partially balanced incomplete block design. *Communications in Statistics - Theory and Methods*, **48**(4), 841-849.

Khattree, R. (2019). On construction of constant block-sum partially balanced incomplete block designs. *Communications in Statistics – Theory and Methods*, **49**(11), 2585-2606.

Khattree, R. (2022). On construction of equi-replicated constant block sum designs. *Communication in Statistics-Theory and Methods*, **51**(13), 4434-4450.