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Low Volatility Optimal Portfolio Selection: Financial Experiments with Mixture Designs

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SUMMARY

This article assesses the performance of various stock market portfolios using a space-filling mixture design and a model for the *logratios* in the portfolio allocation. Traditional portfolio analysis relies on historical correlations between various returns and adopts various optimization models. However, these methods rely excessively on the assumptions made and often tend to ignore the statistical variability in their optimization procedures. As a result, they may not perform well when implemented on independent future data. Additionally, the constraints and bounds imposed play a crucial role in determining the feasible region and hence the optimal value therein. By integrating a systematic simplex design approach for exploring component mixtures in our portfolio and applying a logratio transformation, we offer a robust framework for analyzing the portfolios and exploring how various portfolios perform relative to each other. An advantage of our method is that we are able to estimate the standard error for each portfolio, and this allows us to incorporate the consideration of volatilities into our decision making. Using publicly available stock market data, we demonstrate the effectiveness of our approach.

Keywords: Financial Experiments; Mixture designs; Portfolio optimization; Space-filling Designs.

AMS Subject Classifications: 62K20, 62K99, 62P05

1. INTRODUCTION

This article is dedicated to the memory of Prof. Prem Narain, a dear friend and a well respected figure in the areas of Statistical/Biometrical/Quantitative/Population genetics, Genomics and Molecular breeding, Bioinformatics, Biostatistics and, Agricultural statistics. While fully aware of his multidirectional influence in science in his early days, one of us (Khattree) got to know Prof. Narain better and at a personal level when upon his retirement, Prof. Narain moved for some time to the South-Eastern Michigan area to be with his son. As a young statistician at that time, this author was definitely in awe of all of Prof. Narain's contributions. As can be seen from the above list of his research areas, Prof. Narain's work forms a bridge between theoretical population genetics, statistical methodology, and applied breeding science, making him a foundational figure in multiple disciplines. With this multidisciplinary talent of Prof. Narain in mind, we thought it would be fitting to honor him with research work that is interdisciplinary in nature. Thus, we illustrate here how the powerful statistical techniques more commonly used in agriculture and biological experimentation can also influence the solving of important problems in a drastically different field of finance.

The selection of a diversified portfolio with low volatility and consisting of a number of assets is an important concern for an investor wishing to optimize her overall return with the consistent performance over time, yet with a risk as small as possible. Clearly high returns and low volatility are mutually conflicting goals,

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and the investor must find a balance between the two. Investors continually face the challenge of allocating limited capital across multiple assets and changing the allocations too frequently only results in more volatility in the portfolio, often with inferior returns. Approaches, such as Markowitz's mean-variance optimization method (Markowitz, 2008) provide a valuable framework for portfolio construction. However, most of these classical approaches typically treat the problem as that of mathematical optimization while ignoring the randomness inherent in the market behavior. Markets have changed considerably since this and similar other theories were developed, asset classes have increased several folds and due to numerous other factors such as online and frequency tradings, quick dissemination of political and financial news and, easy and the instant access to information, markets now experience substantially more volatility than what used to be the case forty years ago. This has necessitated the search for alternative statistical approaches that can account for random variations and provide an efficient approach to prediction. Empirical statistical models supported by solid mathematical underpinning can perhaps do that and employing the intensive computational schemes can ease the computational burden of fitting numerous models which comes with the undertaking of such statistical modeling approaches.

The above defines the main goal of this article. Specifically, we wish to construct a portfolio for a moderate time horizon, say three months in the future. The portfolio should have high return, but not necessarily with the best possible performance, yet with low volatility in the sense that for any three month duration, its performance is consistent with relatively fewer big ups and downs. The problem is not that of asset selection from a large asset-space. Instead, we assume that we already have a set of q assets and given a fixed amount of capital to invest, we would like to find the percentages or equivalently, the proportions to be allocated to each of these assets. However, since each asset would have its own return performance and its own variability, our objective is to identify proportions that ensure, with high probability, an overall superior performance, yet with low variability in the future. We do so by what we will term the *financial experiments* by borrowing ideas from statistical experimental designs.

A good discussion of portfolio optimization theory can be found in Cheng *et al.* (2023), Kocuk and Cornuéjols (2020), Monticeli *et al.* (2023), Manogna and Kulkarni (2025) and Gunjan and Bhattacharyya (2023). The foundational discussion of the mean-variance framework for portfolio optimization, establishing the concept of efficient diversification by balancing expected return and risk, is available in Markowitz (2008). A more comprehensive theoretical and computational discussion of portfolio selection and its applications in capital markets is provided by Markowitz and Todd (2000). Neural network-based approaches may sometimes enhance portfolio performance over traditional methods by improving return prediction and adapting to changing market conditions, and this idea has been promoted by Fernández and Gómez (2007). Machine learning techniques can be integrated with portfolio optimization to improve decision-making under uncertainty, and data-driven models can enhance return predictions and risk estimation. Certain claims in this spirit have been made by Ban, Karoui and Lim (2018). The impact of estimation risk on optimal portfolio selection, especially in emerging markets, is discussed in Nathaphan and Chunhachinda (2010). Also, a Bayesian network model is used by Shenoy and Shenoy (2000) to represent and analyze the joint behavior of portfolio risk and return under uncertainty. From the experimental designs point of view, an excellent discussion of mixture design experiments is available in Cornell (1983). The idea of space filing designs is discussion in Joseph (2016), Gomes *et al.* (2018) and Piepel *et al.* (2019). Since the space of portfolios can be seen as a collection of compositional data with random compositions, it is relevant to also refer to Aitchison (1982), Greenacre (2021) and Greenacre *et al.* (2023).

Most of the approaches in the literature are based on the presentation of the portfolio optimization as a mathematical programming problems where an objective function and several constraints are introduced. Our approach here is model based where relying on past data, we model the returns in the presence of statistical error. We present the problem as an experimental design where we seek a (near) optimum; yet we want to have small prediction error. Different portfolios will have different levels of volatilities and the objective of our search via experimental design is to find one or more such candidates for which the two objectives may coincide. This is a more flexible approach in that unlike the previous approaches, one does not need to introduce constraints which in turn require the artificial specifications of certain parameters, such as upper bounds on variances or lower bounds on average returns. Further, in most earlier approaches the role of random error or market noise is in the background in that those are represented via variances, skewness or other measures and the optimization problem makes use of only these summary measures. That in some way results in a gross simplification of the problem. We instead pose returns of the portfolio in terms of a model and rely on direct prediction. We also evaluate our approach on future data by taking it as the test data. As we shall see later, the evaluation of prediction on test data reaffirms the reliability of prediction. In that sense, this work takes a different line of attack on this important problem and we believe further advances the handling of this important million dollar problem, speaking figuratively as well as literally.

The precise statement of the problem and these ideas of financial experiments are elaborated in Section 2. Since the concepts can best be absorbed by real illustrations, in Section 3, we apply this methodology to a real data set where the assets are chosen as certain market indexes. In real applications, these can be replaced by specific suitably chosen exchange traded funds (ETF) which may track these indexes. Of course, they can as well be a group of certain stocks in which case one may surmise that the portfolio will be more volatile. Elaborate evaluation of the choices made here is discussed in Section 4. Since avoiding a severe loss of capital is more important than any gain of the same magnitude, this asymmetry is considered in Section 5. Our approach is further evaluated on future test data in Section 6. We end Section 7 with certain concluding remarks.

Let $X_1, X_2, \cdots, X_q$ be q assets. Their proportions in a given portfolio are denoted by lowercase letters $x_1, x_2, \cdots, x_q$. The respective returns of these assets for a given period T are indicated by $r_1, r_2, \cdots, r_q$. The realized return of a particular portfolio over the period T is denoted by y while its predicted value is denoted by $\hat{y}$. The slippage or tracking error is therefore $(y-\hat{y})$ (usually called by the more familiar term of *residual* in statistics) and a good portfolio must ensure that not only the actual realized performance y is high, but also that these tracking errors for various periods of length T are small.

2. THE GEOMETRY OF PORTFOLIOS AND THE MODEL FORMULATION

Given q assets, $X_i,\ i = 1, 2, \cdots, q$ in respective proportions $x_i,\ i = 1, 2, \cdots, q$ with corresponding returns $r_i,\ i = 1, 2, \cdots, q$ for the duration of T, the realized return y of the portfolio for this duration is given by

$$y = \sum_{i=1}^{q} x_i r_i. \tag{1}$$

Note that since all proportions must be nonnegative and must add to 1, we have the constraint

$$\sum_{i=1}^{q} x_i = 1, x_i \geq 0.$$

Thus, geometrically speaking, any portfolio allocation must fall within a q–simplex. In order to visualize this, we present this q–simplex in Fig. 1 for $q = 3$ as a triangular slice of a plane in a three dimensional space residing within a cube which has each side of length unity. This triangular slice of the plane or 3–simplex is obtained by joining the three corners of this cube with coordinates $(1,0,0), (0,1,0)$ and $(0,0,1)$. In Fig. 1 we have also shown a number of points within this 3-simplex.

Thus, a q–simplex provides a feasible set of portfolio allocations consisting of q or fewer assets. Vertices of the simplex correspond to portfolios concentrated entirely onto a single asset, representing extreme, non-diversified strategies. Points on the edges correspond to a portfolio consisting of strictly fewer than q assets. Points in the interior of the simplex, in contrast, correspond to various diversified portfolios that distribute the total capital across all assets, reflecting a common investment strategy aimed at reducing risk. The centroid of the q–simplex corresponds to the scenario of equal allocation. By exploring the entire simplex, one can systematically examine the behavior

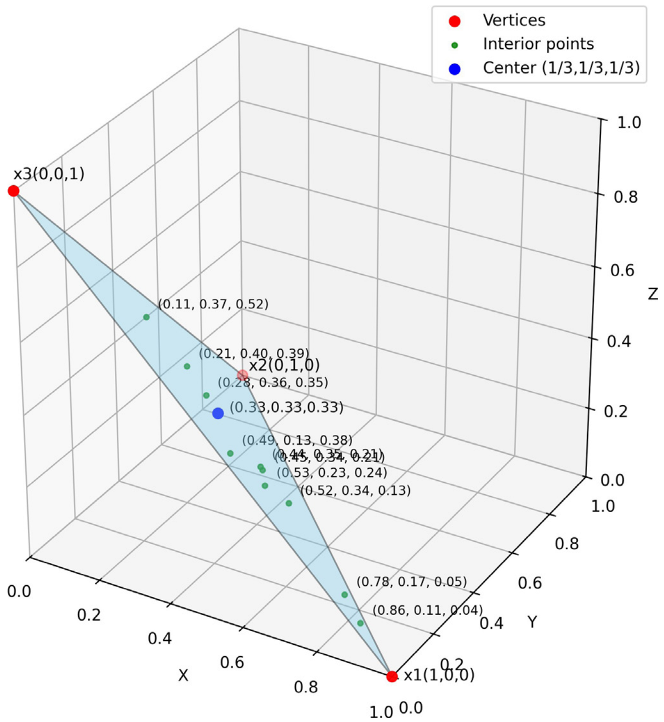

Fig. 1. Visualization of a Three-Component Simplex

and effects of various allocation patterns on the overall portfolio performance as well as on the corresponding risk measured by the average standard error.

Any portfolio must be within q–simplex and as Greenacre (2021) points out, albeit in a different context, when modeling such data, this restriction may invalidate some of the standard statistical analyses or their interpretations. However, by a suitably chosen transformation the q–simplex can be transformed to an infinite space of smaller dimension within this q–dimensional space. One such transformation is the *logratio transformation*. Nonetheless, special issues such as the problems of nonestimability still remain and must be accounted for when interpreting the analyses.

Specifically, the logratio transformation from the *interior of the q*–simplex to the q–dimensional space $\mathbf{x} \rightarrow \mathbf{w}$ is defined by,

$$w_i = log\left(\frac{x_i}{G}\right) = log(x_i) - log(G)$$
$$= log(x_i) - \overline{\log(x)}, \qquad i = 1, 2, \cdots, q, \qquad (2)$$

where $G = (\prod_{i=1}^{q} x_i)^{\frac{1}{q}}$ is the geometric mean of $x_1, x_2, \ldots, x_q$ and

$$\overline{\log(x)} = \frac{1}{q}\sum_{i=1}^{q} log(x_i) = log(\prod_{i=1}^{q} x_i)^{\frac{1}{q}} = log(G).$$

Note that the geometric mean G is defined only for positive allocations. Thus, it is imperative that the points $\mathbf{x}$ are *within the interior of the simplex*. In essence, we have thus assumed here on that $x_i > 0, i = 1, 2, \cdots, q$. It can be seen that $\sum_{i=1}^{q} w_i = 0, w_i \in \mathbb{R}$. Thus, the points $\mathbf{w}$ lie in a $(q-1)$-dimensional subspace within the q–dimensional space defined by above constraint.

We first construct a design within the q–simplex for $\mathbf{x} = (x_1, x_2, \cdots, x_q)'$ for the original asset-proportions. This design is then transformed to another design in terms of $\mathbf{w} = (w_1, w_2, \cdots, w_q)'$. A linear statistical model with portfolio return y as the response, which relates y to $\mathbf{w}$ is then considered with suitable assumptions. Specifically, the model is,

$$y = \beta_0 + \sum_{i=1}^{q} w_i \beta_i + \varepsilon, \varepsilon \sim N(0, \sigma^2). \qquad (3)$$

We assume that the random tracking errors ε are independently and identically distributed. This is an assumption needed for inference (specifically, for the estimation of prediction variances or the standard error of the estimated portfolio returns) and conditions for this will be carefully created via randomization. By construction, for any data set, this model is always *singular* due to the fact that $\sum_{i=1}^{q} w_i = 0$.

3. THE DATA AND THE DESIGN

In order to illustrate the design and the general method, it is perhaps easier to work through a real data set and solve a real problem. This also helps us provide the needed interpretations at various steps and illustrate the methodologies. We thus take $q = 8$ and assume that we are interested in investing in the following eight US-based asset classes (corresponding indexes are used here). The indices corresponding to these asset classes represent a broad spectrum of market sectors, including industrial, energy, services, technology, and precious metals, providing a comprehensive view of market dynamics across different economic segments. These indexes respectively are,

X_1 = Dow Jones Industrials, *DJ-30*

X_2 = DJ US Oil, & Gas, *DJUSEN*

X_3 = Dow Jones US Small Cap, *DJUSS*

X_4 = Nasdaq 100 Index, *NDX–X*

X_5 = NYSE Energy Index, *NYE–X*

X_6 = NYSE International 100, *NYI–X*

X_7 = Standard & Poor's 500, *SP-500*

X_8 = PHLX Gold Silver Sector Total Return Index, *XXAU*.

We consider the data on the daily returns for the above eight indexes over a continuous 14-year period (July 5, 2011 - June 13, 2024). The extended time span ensures that the dataset captures multiple market phases, including periods of expansion and contraction, thereby offering a robust basis for the empirical portfolio modeling. Portfolio returns are computed over the fixed 70-trading days holding periods, which approximates a three month duration. We view this as a moderately long time span and it was chosen to balance the short-term trading noises with longer-term market trends. From an active investor's point of view, such time horizons are more practical than any day-trading scenarios or the year-long scenarios. The daily closing prices are used for calculation of returns and for any given date a period of seventy stock-market-days in the immediate future is used to compute the future performance. In other words, for any selected day t, the corresponding 70-trading days' percent return of the i^{th} asset is given by

$$r_{i,t} = \left(\frac{P_{i,t+70} - P_{i,t}}{P_{i,t}} \right) \times 100 \tag{4}$$

where $P_{i,t}$ and $P_{i,t+70}$ are the closing prices of the i^{th} asset on the starting day t and after 70-trading days, on day $t+70$ respectively.

Table 1. Simplex Design Points

Dsn.Pt.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
1	0.44319	0.00029	0.24361	0.07564	0.05378	0.06696	0.08632	0.03021
2	0.03350	0.00002	0.58358	0.11690	0.02125	0.05485	0.10268	0.08722
3	0.65870	0.00033	0.15386	0.04449	0.00711	0.03108	0.02559	0.07884
4	0.29315	0.00372	0.36549	0.05965	0.12757	0.03579	0.10444	0.01018
5	0.13162	0.02922	0.29032	0.03497	0.13417	0.24153	0.13697	0.00119
6	0.26593	0.10208	0.24511	0.17469	0.09429	0.05279	0.06420	0.00091
7	0.15607	0.01437	0.14454	0.45192	0.07168	0.01428	0.14452	0.00261
8	0.05023	0.00362	0.24727	0.34270	0.18611	0.05714	0.10857	0.00437
9	0.06625	0.00752	0.35752	0.01757	0.42452	0.10942	0.01123	0.00597
10	0.06898	0.00574	0.32218	0.29978	0.08150	0.04906	0.16666	0.00609

Table 2. Logratio Design Points

Dsn.Pt.	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
1	2.25085	−5.06857	1.65240	0.48276	0.14180	0.36094	0.61485	−0.43503
2	0.16189	−7.35610	3.01958	1.41171	−0.29301	0.65506	1.28205	1.11883
3	3.08793	−4.51268	1.63373	0.39303	−1.44107	0.03420	−0.16024	0.96510
4	1.63329	−2.73434	1.85384	0.04117	0.80128	−0.46965	0.60124	−1.72682
5	0.76000	−0.74518	1.55103	−0.56536	0.77918	1.36705	0.79985	−3.94657
6	1.40166	0.44418	1.32010	0.98142	0.36476	−0.21523	−0.01956	−4.27733
7	1.09193	−1.29298	1.01514	2.15511	0.31385	−1.29955	1.01500	−2.99850
8	−0.08137	−2.71203	1.51259	1.83897	1.22843	0.04760	0.68954	−2.52373
9	0.45498	−1.72098	2.14076	−0.87230	2.31253	0.95672	−1.31990	−1.95181
10	0.14936	−2.33635	1.69064	1.61858	0.31614	−0.19150	1.03149	−2.27835

We generate a large space filling design within the q–simplex with $q=8$ by randomly generating a large number of design points uniformly distributed within the *interior of* the simplex*. Clearly each design point corresponds to a specific portfolio allocation. The number of design points generated is taken to be $N_1 = 1000$. Table 1 presents a sample list of only first ten design points from this list. This $N_1 \times q\left(=1000\times 8\right)$ design is then transformed using Equation (2) to a design in terms of $w_1, w_2, \ldots, w_q$. Corresponding ten representative design points are presented in Table 2. The original and the transformed designs consisting of all N_1 design points in their entirety can be found at *https://github.com/Symon7/Logratio_Portfolio.*

Asset returns change from day to day and also intraday. One would normally not buy all assets in the portfolio on the same day. Further, even on the same day, different assets may be purchased at different times. There is also no way to know the *best* day to buy the assets. We incorporate all these uncertainties in our model (3) via a *random error* term ϵ. Thus to make it realistic, one must also inject this random error in our data. To do so we adopt an approach where for the i^{th} design point, $i = 1, 2, \cdots, N_1$,

1. We choose a random date from our specified time horizon (approximately fourteen years in our case).
2. For this selected date, calculate the 70-trading days' forward returns for each asset, using Equation (4).
3. Accordingly compute the return y using Equation (1).

The three steps given above are followed for all design points resulting in our data on portfolio returns $y_1, y_2, \cdots y_{N_1}$. The corresponding $N_1 \times 1$ array of returns $\mathbf{y} = \left[y_1\ y_2\ \cdots\ y_{N_1}\right]'$ is then modeled in terms of $N_1 \times q\left(=1000\times 8\right)$ design matrix $\mathbf{W}$ consisting of various values of $w_1, w_2, \ldots, w_q$.

To set up the notations further, let w_{ji} be the 70-trading days' forward return corresponding to i^{th} asset for the j^{th} design point. Further, the corresponding 70-trading days' forward portfolio return for the j^{th} design point, using Equation (1) is y_j. Then, the matrix version of the model given in Equation (3) for these data can be written as

$$\mathbf{y} = \mathbf{W}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon} \sim \boldsymbol{N}_{N_1}\left(0, \sigma^2 \mathbf{I}_{N_1}\right) \tag{5}$$

where $\mathbf{y} = \left[y_1\ y_2\ \ldots\ y_{N_1}\right]'$, $\boldsymbol{\varepsilon} = \left[\varepsilon_1\ \varepsilon_2\ \ldots\ \varepsilon_{N_1}\right]'$, $\boldsymbol{\beta} = \left[\beta_0\ \beta_1\ \beta_2\ \ldots\ \beta_q\right]'$ and

$$\mathbf{W} = \begin{bmatrix} 1 & w_{11} & w_{12} & \cdots & w_{1q} \\ 1 & w_{21} & w_{22} & \cdots & w_{2q} \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & w_{N_1 1} & w_{N_1 2} & \cdots & w_{N_1 q} \end{bmatrix}.$$

The quantity σ^2 here indicates the unknown error variance, assumed to be constant for all design points. Note that in view of the restriction $\sum_{i=1}^{q} w_{ji} = 0$ for $j = 1, 2, \cdots, N_1$, for each row of matrix $\mathbf{W}$, the last q columns add to zero. This linear dependence makes the matrix $\mathbf{W}$ rank-deficient and hence a *unique* least square estimator of $\boldsymbol{\beta}$ does not exist. Specifically, there are infinitely many least square *solutions*, each specified by a generalized inverse $(\mathbf{W'W})^{-}$ and given by,

$$\hat{\boldsymbol{\beta}} = (\mathbf{W'W})^{-}\mathbf{W'y}. \tag{6}$$

The corresponding predicted values of $\mathbf{y}$ are given by

$$\hat{\mathbf{y}} = \mathbf{W}\hat{\boldsymbol{\beta}} = \mathbf{W}(\mathbf{W'W})^{-}\mathbf{W'y}. \tag{7}$$

It must however be emphasized that even though the least square solution $\hat{\boldsymbol{\beta}}$ is not unique, the values $\hat{\mathbf{y}} = \mathbf{W}\hat{\boldsymbol{\beta}}$ are still unique for all our design points. It is so because the linear functions $\mathbf{W}\hat{\boldsymbol{\beta}}$ are *estimable* and thus their values remain the same irrespective of the choice of $\hat{\boldsymbol{\beta}}$ used. See, Khattree and Naik (1999) for details. In other words, predicted returns for any design point will be the same regardless of which choice of $\hat{\boldsymbol{\beta}}$ is adopted. This is an important and desirable property, and it works to our advantage if our design points are chosen in such a way that the entire space of all possible compositions of portfolios is adequately covered. By considering a large number of design points in our

* This is done by using *SAS IML* procedure and employing the *SAS* function *RandUniSimplex* (N_1, q)

model and in a space filling way, this objective has been accomplished in our work.

As mentioned earlier, since $\mathbf{W}\boldsymbol{\beta}$ is estimable, its estimate does not depend on the choice of a generalized inverse and thus for a given dataset, the vector $\hat{\mathbf{y}}\left(=\mathbf{W}\hat{\boldsymbol{\beta}}\right)$ is uniquely determined even though $\hat{\boldsymbol{\beta}}$ itself is not. The variance–covariance matrix of these predicted values is:

$$Var\left(\hat{\mathbf{y}}\right)=Var\left(\mathbf{W}\hat{\boldsymbol{\beta}}\right)=\sigma^2\mathbf{W}(\mathbf{W}'\mathbf{W})^-\mathbf{W}'$$

Thus, the standard error of $\hat{y}_i,\ i=1,2,\cdots,N_1$ is given by,

$$SE\left(\hat{y}_i\right)=\sqrt{\sigma^2[\mathbf{W}(\mathbf{W}'\mathbf{W})^-\mathbf{W}']_{ii}}$$

and its estimate is given by

$$\hat{S}E\left(\hat{y}_i\right)=\sqrt{\hat{\sigma}^2[\mathbf{W}(\mathbf{W}'\mathbf{W})^-\mathbf{W}']_{ii}} \qquad (8)$$

where $\hat{\sigma}^2=MSE=\dfrac{\mathbf{y}'(\mathbf{I}_{N_1}-\mathbf{W}\left(\mathbf{W}'\mathbf{W}\right)^-\mathbf{W}')\mathbf{y}}{N_1-q}$.

For illustration, we display here in Fig. 2 an analysis for just one random simulation (done using $SAS^{®}$ software).

The $F-test$ statistic examines the null hypothesis that all regression coefficients are zero ($H_0:\beta_1=\beta_2=\ldots=\beta_8=0$ against the alternative hypothesis H_a: At least one of the β_i is non zero). At The corresponding p-value ($=0.0142$) indicates that the model is statistically significant at $\alpha=0.05$. The corresponding prediction equation is given in Equation (9) below.

$$\begin{aligned}\hat{y}=2.30560+0.85314w_1+0.49403w_2+\\ 1.11817w_3+0.81997w_4+0.72404w_5+\\ 0.55550w_6+0.47639w_7+0w_8.\end{aligned} \qquad (9)$$

The coefficient of w_8 in this equation is 0 because the model is singular. It corresponds to just one of the infinitely many solutions. Nonetheless, $\hat{y}$ values for all N_1 portfolios (design points) considered here are invariant of the choice of the least square solution $\hat{\boldsymbol{\beta}}$. The estimated standard error of $\hat{y}$ can be calculated by using the Equation (8).

We simulate the above process $N_2=1000$ times, each time sampling various days and corresponding returns. For each of the N_1 design points, there are thus N_2 simulated predicted values $\hat{y}$ and their corresponding estimated standard errors are given by Equation (8). With self explanatory and obvious notations, averaging these over all N_2 simulations, one correspondingly obtains,

Analysis of Variance					
Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
Model	7	1290.53976	184.36282	2.52	0.0142
Error	992	72529	73.11391		
Corrected Total	999	73820			

Root MSE	8.55067	R-Square	0.0175
Dependent Mean	2.30861	Adj R-Sq	0.0105
Coeff Var	370.38123		

Note: Model is not full rank. Least-squares solutions for the parameters are not unique. Some statistics will be misleading. A reported DF of 0 or B means that the estimate is biased.

Note: The following parameters have been set to 0, since the variables are a linear cnmbination of other variables as shown,

w8 =	-w1 - w2 - w3 - w4 - w5 - w6 - w7

Parameter Estimates					
Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	2.30560	0.27207	8.47	<.0001
w1	B	0.85314	0.30430	2.80	0.0052
w2	B	0.49403	0.29886	1.65	0.0986
w3	B	1.11817	0.30092	3.72	0.0002
w4	B	0.81997	0.30435	2.69	0.0072
w5	B	0.72404	0.31001	2.34	0.0197
w6	B	0.55550	0.29982	1.85	0.0642
w7	B	0.47639	0.29566	1.61	0.1074
w8	0	0	.	.	.

Fig. 2. Analysis of Variance for One Typical Simulation

$$\overline{\hat{y}_{i(.)}}=\frac{1}{N_2}\sum_{j=1}^{N_2}\hat{y}_{i(j)},\text{for}\quad i=1,2,\ldots,N_1 \qquad (10)$$

and

$$\overline{\widehat{\mathrm{SE}}(\hat{y}_{i(.)})}=\frac{1}{N_2}\sum_{j=1}^{N_2}\widehat{\mathrm{SE}}\left(\hat{y}_{i(j)}\right),\text{for}\quad i=1,2,\ldots,N_1. \qquad (11)$$

The value of $\overline{\hat{y}_{i(.)}}$ and $\overline{\widehat{\mathrm{SE}}(\hat{y}_{i(.)})}$ for the first ten design points are given in Table 3. The full details for all $N_1=1000$ points can be found at the link *https://*

Table 3. Simplex Design Points with Corresponding $\overline{\hat{y}_{i(.)}}$ & $\overline{\widehat{\mathrm{SE}}\left(\hat{y}_{i(.)}\right)}$

Dsn.Pt.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{\hat{y}_{i(.)}}$	$\overline{\widehat{\mathrm{SE}}\left(\hat{y}_{i(.)}\right)}$
1	0.44319	0.00029	0.24361	0.07564	0.05378	0.06696	0.08632	0.03021	3.10584	1.71573
2	0.03350	0.00002	0.58358	0.11690	0.02125	0.05485	0.10268	0.08722	3.48230	2.43911
3	0.65870	0.00033	0.15386	0.04449	0.00711	0.03108	0.02559	0.07884	3.09272	1.53927
4	0.29315	0.00372	0.36549	0.05965	0.12757	0.03579	0.10444	0.01018	3.04640	1.52239
5	0.13162	0.02922	0.29032	0.03497	0.13417	0.24153	0.13697	0.00119	2.84698	1.49272
6	0.26593	0.10208	0.24511	0.17469	0.09429	0.05279	0.06420	0.00091	3.17598	1.59286
7	0.15607	0.01437	0.14454	0.45192	0.07168	0.01428	0.14452	0.00261	3.45817	1.66285
8	0.05023	0.00362	0.24727	0.34270	0.18611	0.05714	0.10857	0.00437	3.31212	1.60350
9	0.06625	0.00752	0.35752	0.01757	0.42452	0.10942	0.01123	0.00597	2.59505	1.48194
10	0.06898	0.00574	0.32218	0.29978	0.08150	0.04906	0.16666	0.00609	3.36837	1.38834

github.com/Symon7/Logratio_Portfolio as given earlier.

4. HISTORICAL EVALUATION OF OPTIMAL PORTFOLIO SELECTED

In Table 4, we provide a list of only 10 design points out of $N_1 = 1000$ points from above list where $\overline{\hat{y}_{i(.)}}$ values are large and at the same time $\overline{\widehat{\mathrm{SE}}\left(\hat{y}_{i(.)}\right)}$ are reasonably small. Each of these corresponds to a different portfolio allocation and all of these give superior performance as indicated by averaged (over N_2 simulations) mean performance and with low variability as measured by average estimated standard error of $\hat{y}$ (See Equations 10 and 11). For the 1st design point in Table 4, when

X_1 = Dow Jones Industrials, *DJ-30*, $x_1 = 0.07782$,
X_2 = DJ US Oil, & Gas, *DJUSEN*, $x_2 = 0.01043$,
X_3 = Dow Jones US Small Cap, *DJUSS*, $x_3 = 0.40119$,
X_4 = Nasdaq 100 Index, *NDX–X*, $x_4 = 0.26613$,
X_5 = NYSE Energy Index, *NYE–X*, $x_5 = 0.00107$,
X_6 = NYSE International 100, *NYI–X*, $x_6 = 0.039264$,
X_7 = Standard & Poor's 500, *SP-500*, $x_7 = 0.11689$,
X_8 = PHLX Gold Silver Sector Total Return Index, *XXAU*, $x_8 = 0.08721$,

Table 4. Top Ten Design Points and Corresponding $\overline{\hat{y}_{i(.)}}$ & $\overline{\widehat{\mathrm{SE}}\left(\hat{y}_{i(.)}\right)}$

Dsn.Pt.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{\hat{y}_{i(.)}}$	$\overline{\widehat{\mathrm{SE}}\left(\hat{y}_{i(.)}\right)}$
1	0.07782	0.01043	0.40119	0.26613	0.00107	0.039264	0.11689	0.08721	3.54828	1.19080
2	0.03350	0.00002	0.58358	0.11690	0.02125	0.054853	0.10268	0.08722	3.48230	2.43911
3	0.40297	0.03417	0.23270	0.25827	0.01008	0.001597	0.02727	0.03296	3.47822	1.44903
4	0.15607	0.01437	0.14454	0.45192	0.07168	0.014280	0.14452	0.00261	3.45817	1.66285
5	0.02878	0.03374	0.56065	0.17933	0.01179	0.027145	0.14766	0.01091	3.44493	0.93317
6	0.21133	0.21712	0.04432	0.35113	0.00009	0.018419	0.05395	0.10364	3.42779	1.85819
7	0.06898	0.00574	0.32218	0.29978	0.08150	0.049057	0.16666	0.00609	3.36837	1.38834
8	0.05023	0.00362	0.24727	0.34270	0.18611	0.057139	0.10857	0.00437	3.31212	1.60350
9	0.14217	0.03402	0.32470	0.04179	0.02030	0.000143	0.27954	0.15734	3.30836	2.02217
10	0.00374	0.03303	0.36275	0.46838	0.03850	0.020714	0.01243	0.06045	3.29446	1.37490

we calculate the overall 70-trading days' forward actual (future) **realized** return starting from the first business day of each month for all months from July 2011 to June 2024 ($m=156$ months). Note that the 70-trading days periods necessarily overlap. For the first design point in Table 4, these are listed in Table 5. The overall average 70-trading days' return is then computed as $\bar{y}_{overall}=\frac{1}{m}\sum_{j=1}^{m}\tilde{y}_j$, $\tilde{y}_j=\sum_{i=1}^{q}x_i r_{ij}$, where $q=8,\ i=1,2,\ldots,q$ and $j=1,2,\ldots,m(=156)$. Reader may check that for instance, for the first, fifth and tenth design points of Table 4, the overall actual average monthly returns over entire 14-years period are 3.4897%, 3.3946% and 3.8178% respectively. These closely match the predicted returns reported in Table 4. The detailed returns are available at the *url* link given earlier.

5. CONSIDERATION OF ASYMMETRY IN RISK FOR GAINS AND LOSSES

In the previous section, we have considered the standard error as a measure of risk. This is a linear function of $(y_i-\hat{y}_i)^2$ values. Accordingly, it assigns equal penalty to overestimation and underestimation of the same magnitude of average returns (for example, if the actual return was 3% then the two scenarios of estimating it as 1% or 5% respectively are equally penalized). That is somewhat unrealistic as a better than expected return poses no serious objection as far as portfolio performance is concerned but the contrary is not desirable. One may perhaps want a loss function which for the same degree of overestimation penalizes more compared to one when an underestimation of the same size occurs. One such asymmetric loss function for $\delta_i=(y_i-\hat{y}_i)$ can be taken as the $LINEX$ (Linear-Exponential) loss given by,

$$L(\delta_i)=e^{\delta_i}-b\delta_i-1,$$

where the choice of $b<0$ corresponds to our context when overestimation (that is, when $\delta_i=(y_i-\hat{y}_i)<0.$) is more dearly penalized than an underestimation ($\delta_i=(y_i-\hat{y}_i)>0.$) of the same magnitude. Two such $LINEX$ loss functions for different choices of b are shown in Fig. 3. More negative is the value of b, higher is the penalty $L(\delta_i)$. Of course, the choice of b should be made judiciously. Assigning too high a penalty amounts to playing too safe and thus results in choices with relatively inferior returns.

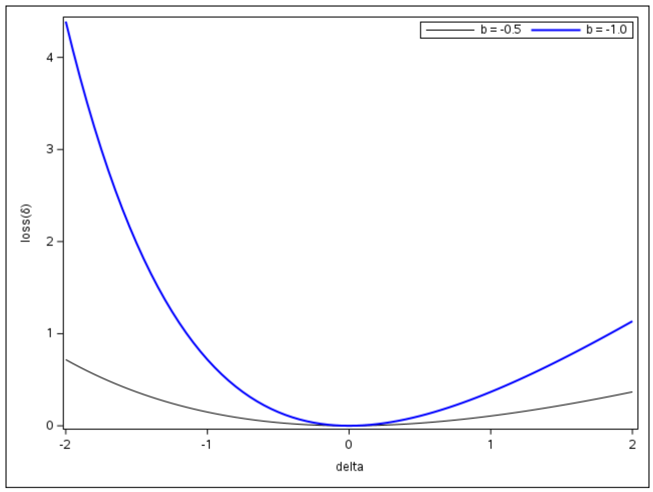

Fig. 3. Two $Linex$ Loss Functions

Table 6 presents the top ten portfolio choices with superior returns, yet with low average $LINEX$ losses each with $b=-0.5$ and $b=-1.0$ respectively (Top ten points are the same for both, while for other later points this may not hold true and rankings may differ for different choices of b). As the table shows, for the first design point and for $b=-0.5$ as well as for $b=-1.0$, the average 70-trading days' return is 2.62%. These translate to the annualized returns of 9.76%. As one anticipates, playing too safe clearly results in a somewhat smaller return. A historic evaluation of these choices are reported in Table 7.

Table 5. Actual Realized 70-trading days' Forward Return on the First Business Day of Each Month for $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (0.0778, 0.0104, 0.4012, 0.2661, 0.0011, 0.0393, 0.1169, 0.0872)$

2011

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	07/05	−8.9573	−15.6611	−16.9180	−2.2059	−16.1889	−16.8977	−10.3507	−3.1295	−10.3984
2	08/01	0.0341	−6.0699	−5.1830	1.5939	−5.5441	−9.4669	−1.5486	3.8734	−1.9367
3	09/01	3.9379	0.0712	0.9207	2.5769	−0.0873	−6.9498	1.9438	−11.2571	0.3349
4	10/03	16.2554	22.5225	22.7214	13.2236	22.2597	10.6540	17.0031	6.3676	17.1196
5	11/01	10.0523	10.0037	12.9329	11.6512	10.3686	6.8360	10.5017	1.3968	10.8048
6	12/01	9.9513	6.3620	12.3113	17.3689	6.5141	7.6071	12.1188	−14.0872	10.8960
Mean										4.4700252

2012

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/03	5.1066	−0.7070	7.8275	16.5349	2.2953	3.2576	8.8167	−7.0652	8.4756
2	02/01	1.1598	−4.7577	1.5529	5.2823	−7.0965	−5.6810	2.7765	−24.3139	0.0429
3	03/01	−4.2075	−15.0320	−6.2804	−4.2565	−16.7164	−14.7764	−4.2587	−17.9766	−6.8002
4	04/02	−5.0286	−9.1320	−5.1954	−8.1764	−10.0105	−11.0336	−5.6285	−17.6865	−7.3912
5	05/01	−0.3762	0.0998	−1.1929	−0.2765	−1.0389	−2.7442	0.2200	−5.2562	−1.1220
6	06/01	9.4630	17.4974	13.5622	13.0763	17.3727	15.9818	11.8912	15.1184	13.1944
7	07/02	4.1588	8.9725	3.5956	4.5867	8.1036	5.9002	5.5278	19.0110	5.6248
8	08/01	−1.2984	−0.3537	3.7423	−1.8845	0.6802	5.4342	0.2796	22.7925	3.1296
9	09/04	1.0894	0.4695	3.1380	−4.5261	0.7722	8.4292	1.0970	−2.6442	0.3735
10	10/01	0.1651	−0.0349	7.0523	−2.1716	0.6767	8.5515	1.7205	−14.7252	1.5173
11	11/01	6.4636	8.6079	12.3007	4.0619	4.8511	7.9753	7.3453	−17.8799	6.2263
12	12/03	11.6601	12.0650	14.8744	4.5846	7.5629	6.0180	10.4568	−19.3548	7.9995
Mean										2.6058762

2013

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	11.4030	4.01400	7.91940	2.9485	−1.34624	−2.61546	8.84104	−36.4555	2.64113
2	02/01	8.8649	3.47648	8.23475	8.8023	0.25082	0.33134	9.05407	−30.0215	4.82581
3	03/01	8.2402	3.59381	7.18304	8.7924	1.88196	−0.16016	8.08802	−22.3706	4.89060
4	04/01	5.0058	2.80692	6.23753	6.8583	0.27456	−1.15541	5.72292	−35.8621	2.24280
5	05/01	4.4910	5.91639	9.91300	8.4043	3.17699	0.13651	6.62375	−9.4931	6.57996
6	06/03	0.4667	3.22524	6.17747	7.0637	2.86736	4.03408	3.44976	−9.1865	4.19170
7	07/01	−1.2887	3.51196	4.56238	6.6147	4.18218	6.10206	2.27919	−2.9980	3.77611
8	08/01	0.4870	3.54151	2.31187	6.8596	3.46910	5.14987	3.44556	−6.3830	2.87987
9	09/03	7.1069	4.60975	8.91459	12.5854	4.22629	5.70885	8.97999	−20.0999	7.05238
10	10/01	8.3961	4.91614	8.06624	9.7963	4.45523	4.57840	8.93899	−6.1292	7.24272
11	11/01	2.0588	−1.71504	4.09276	7.0203	−1.84515	−0.30564	3.22474	4.4265	4.40163
12	12/02	0.3801	0.01511	4.81181	4.3194	0.10895	−1.29942	2.27788	29.0232	5.85616
Mean										4.7150733

2014

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	−2.36470	1.8881	−0.90830	−3.10638	3.2997	0.7548	−0.65327	10.4918	−0.48363
2	02/03	8.06731	15.7539	4.89749	4.42795	16.6036	9.9950	8.39877	1.5218	5.46000
3	03/03	4.67045	10.6972	2.10927	4.01191	10.5617	5.5446	5.75871	−11.5469	2.28407
4	04/01	2.44689	9.4039	0.53698	7.33891	5.9015	3.3745	4.57160	12.0869	4.18432
5	05/01	0.19376	2.8453	1.55521	8.92034	−6.7962	−0.4966	2.94935	11.6164	4.37371
6	06/02	1.75338	−1.1352	2.76767	9.35087	−1.4879	0.5236	3.49743	9.3647	4.96799
7	07/01	−1.06658	−15.1420	−9.16910	2.62849	−14.0765	−8.4037	−1.76987	−22.1142	−5.70044
8	08/01	6.75571	−8.9189	5.76906	7.68101	−5.2092	−5.1737	5.93831	−34.0477	2.30746
9	09/02	2.93391	−23.4216	−0.96448	3.44932	−25.3026	−10.1758	1.47430	−28.0189	−2.18275
10	10/01	4.74156	−16.9593	7.35070	4.66157	−19.3318	−8.7548	4.16435	−5.6024	4.01546
11	11/03	3.58659	−6.8585	5.04099	4.67260	−7.0875	−0.8094	3.62620	19.6150	5.56862
12	12/01	−0.54528	−6.2901	4.84827	0.34201	−8.7901	−5.2506	−0.40984	−8.0889	0.95915
Mean										2.1461628

2015

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	1.7732	2.7783	6.4587	4.7446	2.7833	5.8398	2.5307	3.3994	4.8454
2	02/02	5.8778	5.7925	6.4317	7.9864	5.0802	7.8626	5.8125	−4.7418	5.8036
3	03/02	−1.9711	−0.5975	0.4975	−0.3498	−0.3930	−1.0248	−1.1110	−8.6166	−0.9751
4	04/01	1.1502	−5.2087	−0.0690	4.0261	−4.8728	0.0982	1.5451	−12.6035	0.1591
5	05/01	−2.8368	−18.0828	−1.6951	1.2586	−17.5714	−4.8715	−0.5217	−33.8164	−3.9747
6	06/01	−9.7984	−19.5667	−8.0901	−5.4744	−19.6153	−10.8237	−7.8315	−35.8747	−10.1592
7	07/01	−3.4635	−7.4690	−6.7218	−1.5401	−6.8930	−5.6589	−2.8732	−10.4404	−4.9299
8	08/03	0.9208	0.9448	−4.9759	1.4751	−1.3504	−7.8565	−0.8336	0.7249	−1.8663
9	09/01	9.3537	−2.8013	−0.5642	12.0857	−3.3681	−0.1358	7.4932	0.7643	4.6223
10	10/01	1.5444	−11.7788	−6.5838	3.5046	−12.7719	−6.3377	0.7182	−6.7181	−2.4758
11	11/02	−11.1209	−20.2690	−14.3987	−14.7602	−20.0558	−18.0126	−11.8880	14.2405	−11.6580
12	12/01	−3.1522	−9.8682	−5.2730	−7.1512	−8.8370	−6.9269	−3.4029	44.2994	−1.1827
Mean										−1.8159502

2016

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/04	5.4713	8.1538	5.4386	2.3097	9.7909	1.83481	4.4454	64.6720	9.5496
2	02/02	9.4171	18.9163	11.7461	3.1649	19.9495	5.44923	8.2138	88.8661	15.4302
3	03/02	6.2388	17.2679	8.4875	3.4001	16.2677	4.77293	6.1540	46.5987	9.9636
4	04/01	3.9339	11.9242	6.8348	2.6605	14.6089	4.24770	4.6732	57.7731	9.6476
5	05/02	3.9107	−0.4996	6.0362	10.0907	−0.9621	0.43764	5.1251	24.7135	8.1767
6	06/01	2.3813	4.2026	3.7167	3.9700	4.9608	3.79582	2.0483	17.5901	4.7046
7	07/01	0.8096	1.8277	2.6859	8.8962	1.2455	2.90534	1.4748	−19.5548	2.1096
8	08/01	−0.8444	4.2943	3.5904	0.6029	4.0917	−0.98798	−1.9689	−21.3192	−0.5439
9	09/01	7.9378	12.2165	7.1177	2.1900	11.5797	1.54116	4.4171	−10.2306	3.8805
10	10/03	8.6066	6.0833	7.2786	3.0292	7.0678	2.25322	4.6167	−6.2692	4.5484
11	11/02	13.6189	8.0587	17.2973	11.0466	7.5563	8.58214	10.8964	7.1390	13.2646
12	12/01	8.9912	−7.6382	5.5608	13.8097	−5.8845	7.43355	8.2974	−0.0954	7.7733
Mean										7.3754039

2017

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/03	3.4250	−8.4376	1.73951	9.6032	−7.1264	3.35764	3.73341	12.7258	5.10252
2	02/01	5.1585	−6.5170	1.90048	10.5896	−4.6838	5.31574	5.04964	−6.3986	4.15006
3	03/01	0.9646	−9.6466	−0.68608	5.6463	−7.6968	4.34961	1.49446	2.4024	1.74860
4	04/03	4.8472	−7.4433	2.19511	6.9089	−6.0870	6.90520	4.13523	−5.0966	3.32240
5	05/01	5.2555	−4.7703	0.27012	4.7472	−2.3602	5.98408	3.23922	2.6611	2.57412
6	06/01	4.4468	−1.9671	1.87186	3.1312	−0.1174	4.91891	2.52658	8.7123	3.15795
7	07/03	6.6846	5.1352	4.60933	8.2236	7.1092	5.79974	4.91047	8.6081	6.17156
8	08/01	7.1545	5.6497	3.66777	7.2812	7.1382	1.96897	4.60840	−2.5241	4.42841
9	09/01	11.2352	11.5111	7.10335	6.6879	10.7819	3.16588	7.51582	−14.5355	5.37075
10	10/02	13.2949	11.5855	6.99253	11.9028	10.5757	5.96048	9.21997	2.7655	8.69272
11	11/01	4.4129	−1.6694	0.26081	4.0779	−2.4124	−0.10224	2.41406	−0.6736	1.73271
12	12/01	3.4754	−3.4003	4.10364	12.2528	−3.5621	1.04077	5.21777	−0.4559	5.74938
Mean										4.3500997

2018

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	−2.0126	−2.0198	−1.1619	2.8139	−0.5522	−2.01207	−1.3909	−3.0583	−0.4039
2	02/01	−4.4280	3.3284	−0.5329	1.0993	4.9223	−4.76879	−3.1013	−3.8137	−1.1082
3	03/01	3.4679	16.4257	8.7530	6.8584	15.0997	0.47361	4.5314	7.7423	7.0177
4	04/02	5.6514	16.7212	10.0850	14.1972	14.9197	−0.17724	8.4960	−0.8145	9.3696
5	05/01	7.0753	3.6356	9.3578	13.3821	2.0638	0.13592	8.6299	−7.0623	8.3046
6	06/01	4.9394	−4.6076	4.3728	5.6178	−4.8102	−3.12168	5.4467	−26.4328	1.7895
7	07/02	6.2967	1.0911	−1.5716	1.0460	0.7774	−0.81743	3.1831	−20.7633	−1.3207
8	08/01	3.1789	−9.6088	−3.9686	−1.5136	−9.1650	−6.46468	−0.3870	−14.0357	−3.3807
9	09/04	−5.1642	−16.0783	−14.8416	−11.3081	−14.4721	−7.64696	−8.5909	8.3807	−10.1224
10	10/01	−10.5199	−20.0203	−12.7862	−13.9683	−18.2936	−8.99819	−11.6697	10.6256	−10.6849
11	11/01	0.7965	−5.0291	3.4953	0.5407	−3.8969	0.63148	0.8262	13.0116	2.8076
12	12/03	0.4486	−0.7976	3.1978	4.0017	0.5640	4.06158	1.7527	15.9072	4.1267
Mean										0.5329058

2019

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	14.7463	20.1050	21.8037	22.5514	18.9169	14.3940	17.4632	8.4930	19.4737
2	02/01	1.6073	−1.7993	2.8720	7.0090	−0.9951	0.6420	4.5697	−9.0820	2.8901
3	03/01	0.3261	−6.4606	−1.0087	5.4769	−5.2840	0.6634	3.2701	−1.0657	1.3206
4	04/01	3.2396	−4.3740	1.5343	6.0724	−3.4762	1.3364	4.9206	13.0707	4.2019
5	05/01	−1.2440	−10.4825	−2.7418	−2.0465	−11.2138	−4.6617	−0.7939	38.5022	1.2192
6	06/03	8.9338	1.9478	5.5960	12.6197	−1.5810	3.3759	9.0332	30.1200	10.1326
7	07/01	−1.3768	−12.9769	−4.5186	−1.0366	−12.1668	−4.6358	−1.5178	12.5226	−1.6116
8	08/01	3.8806	−2.0124	2.6335	5.3712	0.8451	4.3265	4.3579	6.0497	3.9747
9	09/03	7.0196	3.9311	7.7989	10.2382	6.4598	8.3393	8.3462	−2.2605	7.5536
10	10/01	8.3836	6.1473	8.9580	16.6298	6.7527	9.6613	10.9627	15.8383	11.7853
11	11/01	8.1162	−4.0231	6.6120	17.5180	−4.1342	5.0151	10.1550	8.8303	10.0540
12	12/02	−23.3853	−50.6019	−29.5760	−11.7555	−48.5921	−25.3600	−19.8796	−26.4071	−23.0161
Mean										3.9981706

2020

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	−17.2711	−44.8229	−24.9950	−3.5981	−42.1996	−21.0806	−13.3032	−1.8441	−15.3856
2	02/03	−18.5455	−33.4829	−24.4459	−1.5603	−35.0776	−20.9909	−13.6760	13.2562	−13.3194
3	03/02	6.0889	−4.5949	1.4483	18.6579	−9.4019	−1.0652	8.0233	21.6612	8.7474
4	04/01	25.3064	29.2936	31.9343	42.4381	17.5152	20.5832	28.6800	68.6000	36.5426
5	05/01	16.8288	7.9461	23.2594	24.9348	9.9264	17.2252	17.8916	27.7860	22.5613
6	06/01	9.8486	−11.4994	6.5657	17.6745	−8.3207	9.7559	11.0604	24.7606	11.8107
7	07/01	10.6054	−15.5080	13.1896	14.3758	−13.3512	10.3512	11.5086	17.4039	13.0361
8	08/03	9.7708	−7.5136	10.7407	4.6809	−5.3039	6.5398	6.9228	−5.4017	6.8260
9	09/01	5.6064	14.9474	16.3401	0.7694	13.9155	8.6440	4.3089	−7.0745	7.5934
10	10/01	11.6357	45.5622	28.5521	11.2730	43.2952	17.0781	12.3497	−0.5311	17.9497
11	11/02	17.4458	50.9228	31.9717	24.6385	46.6402	23.9315	19.0881	−1.2074	24.3880
12	12/01	9.5050	41.8682	17.7302	4.4356	35.3673	9.1791	7.6160	3.0860	11.0276
Mean										10.9814938

2021

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/04	13.0739	28.8827	16.6011	11.1154	23.1604	8.05119	13.0245	−3.0299	12.5361
2	02/01	11.7949	33.0073	9.0612	−0.1226	26.4386	6.56766	8.8914	7.4419	6.8393
3	03/01	10.8591	16.0292	6.5660	5.8180	15.2587	8.16446	9.7913	23.6540	8.7391
4	04/01	5.7345	7.1858	3.9300	11.9795	6.7819	5.40883	9.3704	3.0239	6.8646
5	05/03	4.1174	0.7762	1.0830	8.6178	2.3894	3.15679	5.9310	−7.1180	3.2555
6	06/01	0.8813	−8.4677	2.3144	14.3700	−5.8974	−2.31156	7.0155	−20.7218	3.6489
7	07/01	−0.0358	5.1046	−0.7001	1.5579	6.8382	−3.62892	1.3919	−11.5984	−0.7998
8	08/02	3.9211	19.6329	6.4087	8.1866	17.3475	1.20528	6.5239	−5.2286	5.6322
9	09/01	1.1998	18.2871	−2.1811	3.4308	14.5905	−3.10451	3.2777	−6.1970	0.0585
10	10/01	5.8719	16.5868	0.9376	6.4705	13.2745	6.95198	8.1555	7.7098	4.6409
11	11/01	−1.9477	15.9894	−7.9770	−7.2770	15.9902	5.01620	−2.4102	−0.3663	−5.2213
12	12/01	−3.4958	33.8719	−8.1690	−17.9249	29.3687	1.13476	−7.7279	19.6871	−7.0768
Mean										3.2597643

2022

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/03	−5.8097	39.0050	−8.3919	−14.6392	35.1174	0.8152	−7.6809	27.8701	−5.7056
2	02/01	−10.7203	14.3496	−13.2277	−20.7286	10.3416	−13.8129	−13.9323	−3.4867	−13.9718
3	03/01	−2.5368	27.0641	−6.1906	−11.8003	23.0641	−4.1736	−6.1308	−11.1289	−7.3655
4	04/01	−12.7231	−14.3466	−18.3241	−21.9750	−15.8125	−18.4772	−17.4376	−35.9327	−20.2538
5	05/02	2.5885	2.6228	4.0429	4.3622	2.2595	−0.5257	3.4191	−19.7500	1.6707
6	06/01	−1.3051	−5.2927	2.5397	1.4597	−7.7598	−8.1749	0.2410	−16.8837	−0.5229
7	07/01	−4.6365	13.1779	−1.8967	−5.8206	7.3221	−9.7764	−4.8946	−6.8668	−4.0803
8	08/01	0.5841	21.2289	−5.4066	−15.0830	14.3099	−6.3120	−7.5620	−0.1802	−7.0484
9	09/01	7.3663	5.7343	1.0095	−4.0254	3.9575	6.5435	0.8108	25.6066	2.5559
10	10/03	17.1090	20.3218	12.4153	2.3761	19.3515	20.4021	9.2271	30.1316	11.6847
11	11/01	4.3163	−2.2495	7.8894	9.1784	1.1678	18.4392	6.4746	15.9123	8.7900
12	12/01	−7.9097	−16.5452	−8.4425	0.8175	−14.8383	−4.4085	−5.2490	−3.1959	−5.0388
Mean										−3.2738275

2023

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/03	2.6796	3.7379	2.9670	20.3472	5.5387	8.8591	8.4039	14.0237	9.4119
2	02/01	−1.4020	−10.4468	−8.7360	10.3725	−7.4879	−1.6590	1.5337	1.0053	−0.7686
3	03/01	3.9582	−3.0206	−2.0772	21.6762	−2.6252	1.9168	8.9527	8.1986	7.0458
4	04/03	3.5480	−1.8165	7.3206	19.1343	0.3594	3.9981	9.7778	−2.5217	9.3667
5	05/01	3.1662	5.4054	6.8414	14.3033	5.3448	3.3762	7.0555	−11.6247	6.8032
6	06/01	5.6752	18.6168	7.5283	7.4469	18.0832	7.1140	6.8364	−6.8908	6.1348
7	07/03	−1.9666	6.3696	−5.2045	−0.2242	7.4408	−0.3655	−2.1824	−7.8415	−3.1796
8	08/01	−4.3078	−3.9922	−11.0053	−2.6584	−1.5659	−3.8987	−4.5488	−10.8190	−7.1296
9	09/01	4.7595	−9.7860	−1.4088	4.8356	−6.1762	1.8212	2.3713	−2.5263	1.1118
10	10/02	12.6574	−7.5187	10.3269	13.0418	−4.1932	8.3983	11.2523	13.3798	11.3278
11	11/01	15.2344	−3.6691	17.5682	21.0244	−2.7505	11.4974	17.2133	−5.3097	15.7882
12	12/01	7.7682	6.6445	13.2508	13.1273	4.3709	9.9353	12.4836	−2.6825	11.1035
Mean										5.5846579

2024

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	1.01715	13.7180	3.45571	9.1104	10.4157	6.24252	8.15789	10.2313	6.13526
2	02/01	3.40152	12.9556	8.09014	5.6576	11.1229	8.08806	7.36867	23.0392	8.35122
3	03/01	−1.04055	4.9679	−0.67622	4.9939	4.1637	3.84277	4.57563	29.5565	4.29638
4	04/01	0.33400	−4.2155	−2.73800	10.7281	−2.7687	4.89417	6.64212	17.3573	4.21796
5	05/01	3.89402	−1.6233	2.44344	6.6490	−2.2707	2.95198	6.20172	7.6816	4.54419
6	06/03	6.32284	−6.8531	1.32412	4.2095	−7.1793	3.28300	5.75395	5.0199	3.30365
Mean										5.1414428

Table 6. Linex Loss Design Points with Corresponding $\hat{y}_{i(.)}$ & $L(\delta_{i(.)})$

								$b=-0.5$		$b=-1$	
Dsn. Pt.	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	$\overline{\hat{y}_{i(.)}}$	$L(\delta_{i(.)})$	$\overline{\hat{y}_{i(.)}}$	$L(\delta_{i(.)})$
1	0.09468	0.46247	0.07311	0.04635	0.05321	0.09564	0.16573	0.00881	2.61597	0.000002	2.61597	0.000007
2	0.08561	0.09140	0.16247	0.06613	0.39211	0.08165	0.04535	0.07528	2.47965	0.000003	2.47965	0.000013
3	0.00475	0.14394	0.03630	0.08303	0.33663	0.08242	0.03076	0.28217	2.26521	0.000006	2.26521	0.000024
4	0.03401	0.22781	0.40909	0.05458	0.12179	0.00402	0.07268	0.07602	2.88198	0.000045	2.88198	0.000180
5	0.01537	0.16139	0.04583	0.32678	0.04298	0.08535	0.31174	0.01056	3.00252	0.000046	3.00252	0.000186
6	0.17669	0.26367	0.19770	0.14271	0.12260	0.07602	0.00074	0.01988	2.65529	0.000070	2.65529	0.000281
7	0.11971	0.07478	0.06653	0.00002	0.02072	0.47212	0.23136	0.01475	1.35268	0.000095	1.35268	0.000381
8	0.25797	0.04361	0.21792	0.10843	0.09052	0.07830	0.18204	0.02121	2.89414	0.000156	2.89414	0.000620
9	0.16654	0.12559	0.04914	0.17962	0.08341	0.08746	0.29581	0.01244	2.81413	0.000203	2.81413	0.000805
10	0.07951	0.14211	0.31693	0.04299	0.04518	0.03088	0.25975	0.08264	2.78499	0.000263	2.78499	0.001061

For the 1^{st} design point in Table 6 when

X_1 = Dow Jones Industrials, *DJ-30*, $x_1 = 0.09468$,

X_2 = DJ US Oil, & Gas, *DJUSEN*, $x_2 = 0.46247$,

X_3 = Dow Jones US Small Cap, *DJUSS*, $x_3 = 0.07311$,

X_4 = Nasdaq 100 Index, *NDX–X*, $x_4 = 0.04635$,

X_5 = NYSE Energy Index, *NYE–X*, $x_5 = 0.05321$,

X_6 = NYSE International 100, *NYI–X*, $x_6 = 0.09564$,

X_7 = Standard & Poor's 500, *SP-500*, $x_7 = 0.16573$,

X_8 = PHLX Gold Silver Sector Total Return Index, *XXAU*, $x_8 = 0.00881$,

we compute the 70-trading days' forward realized returns beginning from the first trading day of each month for all months spanning from July 2011 through June 2024. The resulting values are reported in Table 7. In addition, we calculate the overall average 70-trading days' return for the portfolio over the entire period. The reader may verify that the first, fifth, and tenth design points in Table 6 yield the overall actual average monthly returns of 2.60%, 3.58%, and 3.09%, respectively, over the full 14-year period. These values are consistent with the corresponding predicted returns reported in Table 6.

6. MODEL VALIDATION

It is important that we also validate our model and our approach for the out of sample data. To accomplish this, we take the data from June 14, 2024 to December 8, 2025 covering a total of $N_{test} = 372$ trading days and calculate the 70-trading days' forward returns for all eight assets. Accordingly, we consider the portfolio corresponding to the 1^{st} design point in Table 4 namely, $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*, x_8^*) = (0.0778, 0.0104, 0.4012, 0.2661, 0.0011, 0.0393, 0.1169, 0.0872)$ and calculate the 70-trading days' portfolio's realized returns $y_i = \sum_{j=1}^{q} r_{ij} x_j |_{\mathbf{x}=\mathbf{x}^*}$, for $i = 1, 2, ..., N_{test}$. For brevity, these are shown in Table 8 only for $m = 18$ dates starting from the first trading day of each month. Note that the 70-trading days periods thus necessarily overlap. The predicted return $\hat{y} = \sum_{j=1}^{q} \hat{\beta}_j w_j |_{\mathbf{w}=\mathbf{w}^*}$, where $\mathbf{w}^*$ corresponds to the simplex point $\mathbf{x}^*$ given above is equal to 3.12223%.

While for simplicity, Table 8 presents the information only for 18 days, it can also be shown that the overall actual average 70-trading days' forward return for the above portfolio over the entire $N_{test} = 372$ day period is 6.00606% (the predicted return for the same using the above model is 3.12223%). Further, the readers may verify that only for the eighteen dates (the first trading day of the month) listed in Table 8 the average realized 70-trading days' forward return is 6.62276%. The test data successfully validates the usefulness of our model and justifies the utility of our approach.

7. CONCLUDING REMARKS

This work presents a unified financial-statistical framework for portfolio optimization based on mixture designs and compositional data principles. The goal is not to find a portfolio that will outperform just tomorrow. Instead our objective is to choose a portfolio that is robust (low variability and hence less volatile) yet gives superior performance. In that sense, what our framework delivers is a statistically optimal decision in a moderate to long run. By integrating the statistically sound ideas of space-filling mixture designs, compositional data and suitable transformation, one can develop, for a given context, a rigorous methodology to explore how relative asset proportions influence overall portfolio returns. The approach not only adheres to the fundamental structure of financial data where asset proportions should be positive and must sum to one but also introduces a statistically coherent way to estimate, interpret, and optimize portfolio performance.

Our approach is illustrated using the data on eight indexes. To give an overall feel, if we consider the overall seventy day performance for the first design point in Table 4 an estimated 3.55% return roughly translates to a $100 \times \left(1.0355^{\frac{252}{70}} - 1 \right) = 13.38\,\%$ annualized return. From an applied perspective, this framework can easily be extended to more complex investment settings where assets are carefully chosen and portfolio consists of considerably more number of assets.

One may ask: How does this approach compare with other standard approaches? We have addressed this issue and philosophically contrasted our approach with other approaches in the Introduction section. As emphasized there, statistical modeling has its own distinct advantages. Also, as pointed out earlier, most mathematical programming and machine learning approaches impose constraints that require artificial specification of upper and lower bounds for those constraints. That being the case, for the data presented here, our approach cannot be directly compared with these previous approaches, as these bounds can be conveniently manipulated accordingly to show either inferior or superior results. Further, these optimization based approaches concentrate only on very long-run returns. Our model based approach is more practical and suitable for moderate-term returns. Market changes over time, and based on how it evolves, one can refine and update the models for not-so-long-term portfolio selection problems.

One thing should be kept in mind. It is certainly appropriate and perhaps preferred to consider an asymmetric loss function. The $LINEX$ loss is one such choice but many similar losses can be devised. However, with such a selection, one must also be mindful that overestimation penalty is not excessively severe. That would amount to taking either too little or absolutely no risk. It then defeats the whole purpose of portfolio selection and when trading costs are considered, it may not be worth going through such an extensive exercise for a very little gain in the portfolio.

Supplementary Information, Conflict of Interest Statement and Acknowldgement

The raw data for this work were obtained by using the software $TC2000$, $Version 7$. Detailed analyses and data are available at *https://github.com/Symon7/Logratio_Portfolio*. Both authors declare that there was no conflict of interest with any parties pertaining to this work. Both authors wish to thank a referee and an associate editor for their critical comments.

Table 7. Actual Realized 70-trading Days' Forward Return on the First Business Day of Each Month for $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (0.09468, 0.46247, 0.07311, 0.04635, 0.05321, 0.09564, 0.16573, 0.00881)$

2011

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	07/05	−8.9573	−15.6611	−16.9180	−2.2059	−16.1889	−16.8977	−10.3507	−3.1295	−13.6505
2	08/01	0.0341	−6.0699	−5.1830	1.5939	−5.5441	−9.4669	−1.5486	3.8734	−4.5319
3	09/01	3.9379	0.0712	0.9207	2.5769	−0.0873	−6.9498	1.9438	−11.2571	0.1461
4	10/03	16.2554	22.5225	22.7214	13.2236	22.2597	10.6540	17.0031	6.3676	19.3065
5	11/01	10.0523	10.0037	12.9329	11.6512	10.3686	6.8360	10.5017	1.3968	10.0220
6	12/01	9.9513	6.3620	12.3113	17.3689	6.5141	7.6071	12.1188	−14.0872	8.5480
Mean										3.30671

2012

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/03	5.1066	−0.7070	7.8275	16.5349	2.2953	3.2576	8.8167	−7.0652	3.3278
2	02/01	1.1598	−4.7577	1.5529	5.2823	−7.0965	−5.6810	2.7765	−24.3139	−2.4071
3	03/01	−4.2075	−15.0320	−6.2804	−4.2565	−16.7164	−14.7764	−4.2587	−17.9766	−11.1735
4	04/02	−5.0286	−9.1320	−5.1954	−8.1764	−10.0105	−11.0336	−5.6285	−17.6865	−8.1347
5	05/01	−0.3762	0.0998	−1.1929	−0.2765	−1.0389	−2.7442	0.2200	−5.2562	−0.4171
6	06/01	9.4630	17.4974	13.5622	13.0763	17.3727	15.9818	11.8912	15.1184	15.1424
7	07/02	4.1588	8.9725	3.5956	4.5867	8.1036	5.9002	5.5278	19.0110	7.0978
8	08/01	−1.2984	−0.3537	3.7423	−1.8845	0.6802	5.4342	0.2796	22.7925	0.7028
9	09/04	1.0894	0.4695	3.1380	−4.5261	0.7722	8.4292	1.0970	−2.6442	1.3457
10	10/01	0.1651	−0.0349	7.0523	−2.1716	0.6767	8.5515	1.7205	−14.7252	1.4237
11	11/01	6.4636	8.6079	12.3007	4.0619	4.8511	7.9753	7.3453	−17.8799	7.7612
12	12/03	11.6601	12.0650	14.8744	4.5846	7.5629	6.0180	10.4568	−19.3548	10.5241
Mean										2.09943

2013

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	11.4030	4.01400	7.91940	2.9485	−1.34624	−2.61546	8.84104	−36.4555	4.47391
2	02/01	8.8649	3.47648	8.23475	8.8023	0.25082	0.33134	9.05407	−30.0215	4.73820
3	03/01	8.2402	3.59381	7.18304	8.7924	1.88196	−0.16016	8.08802	−22.3706	4.60306
4	04/01	5.0058	2.80692	6.23753	6.8583	0.27456	−1.15541	5.72292	−35.8621	3.08259
5	05/01	4.4910	5.91639	9.91300	8.4043	3.17699	0.13651	6.62375	−9.4931	5.47187
6	06/03	0.4667	3.22524	6.17747	7.0637	2.86736	4.03408	3.44976	−9.1865	3.34399
7	07/01	−1.2887	3.51196	4.56238	6.6147	4.18218	6.10206	2.27919	−2.9980	3.29976
8	08/01	0.4870	3.54151	2.31187	6.8596	3.46910	5.14987	3.44556	−6.3830	3.36283
9	09/03	7.1069	4.60975	8.91459	12.5854	4.22629	5.70885	8.97999	−20.0999	6.12188
10	10/01	8.3961	4.91614	8.06624	9.7963	4.45523	4.57840	8.93899	−6.1292	6.21469
11	11/01	2.0588	−1.71504	4.09276	7.0203	−1.84515	−0.30564	3.22474	4.4265	0.47241
12	12/02	0.3801	0.01511	4.81181	4.3194	0.10895	−1.29942	2.27788	29.0232	1.10970
Mean										3.8579060

2014

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	−2.36470	1.8881	−0.90830	−3.10638	3.2997	0.7548	−0.65327	10.4918	0.6708
2	02/03	8.06731	15.7539	4.89749	4.42795	16.6036	9.9950	8.39877	1.5218	11.8575
3	03/03	4.67045	10.6972	2.10927	4.01191	10.5617	5.5446	5.75871	−11.5469	7.6744
4	04/01	2.44689	9.4039	0.53698	7.33891	5.9015	3.3745	4.57160	12.0869	6.4610
5	05/01	0.19376	2.8453	1.55521	8.92034	−6.7962	−0.4966	2.94935	11.6164	2.0434
6	06/02	1.75338	−1.1352	2.76767	9.35087	−1.4879	0.5236	3.49743	9.3647	0.9098
7	07/01	−1.06658	−15.1420	−9.16910	2.62849	−14.0765	−8.4037	−1.76987	−22.1142	−9.6931
8	08/01	6.75571	−8.9189	5.76906	7.68101	−5.2092	−5.1737	5.93831	−34.0477	−2.7951
9	09/02	2.93391	−23.4216	−0.96448	3.44932	−25.3026	−10.1758	1.47430	−28.0189	−12.7867
10	10/01	4.74156	−16.9593	7.35070	4.66157	−19.3318	−8.7548	4.16435	−5.6024	−7.8659
11	11/03	3.58659	−6.8585	5.04099	4.67260	−7.0875	−0.8094	3.62620	19.6150	−1.9279
12	12/01	−0.54528	−6.2901	4.84827	0.34201	−8.7901	−5.2506	−0.40984	−8.0889	−3.6994
Mean										−0.7625918

2015

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	1.7732	2.7783	6.4587	4.7446	2.7833	5.8398	2.5307	3.3994	3.3009
2	02/02	5.8778	5.7925	6.4317	7.9864	5.0802	7.8626	5.8125	−4.7418	6.0196
3	03/02	−1.9711	−0.5975	0.4975	−0.3498	−0.3930	−1.0248	−1.1110	−8.6166	−0.8217
4	04/01	1.1502	−5.2087	−0.0690	4.0261	−4.8728	0.0982	1.5451	−12.6035	−2.2232
5	05/01	−2.8368	−18.0828	−1.6951	1.2586	−17.5714	−4.8715	−0.5217	−33.8164	−10.4822
6	06/01	−9.7984	−19.5667	−8.0901	−5.4744	−19.6153	−10.8237	−7.8315	−35.8747	−14.5148
7	07/01	−3.4635	−7.4690	−6.7218	−1.5401	−6.8930	−5.6589	−2.8732	−10.4404	−5.8211
8	08/03	0.9208	0.9448	−4.9759	1.4751	−1.3504	−7.8565	−0.8336	0.7249	−0.7263
9	09/01	9.3537	−2.8013	−0.5642	12.0857	−3.3681	−0.1358	7.4932	0.7643	1.1654
10	10/01	1.5444	−11.7788	−6.5838	3.5046	−12.7719	−6.3377	0.7182	−6.7181	−6.8459
11	11/02	−11.1209	−20.2690	−14.3987	−14.7602	−20.0558	−18.0126	−11.8880	14.2405	−16.7982
12	12/01	−3.1522	−9.8682	−5.2730	−7.1512	−8.8370	−6.9269	−3.4029	44.2994	−6.8855
Mean										−4.5527707

2016

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/04	5.4713	8.1538	5.4386	2.3097	9.7909	1.83481	4.4454	64.6720	6.7966
2	02/02	9.4171	18.9163	11.7461	3.1649	19.9495	5.44923	8.2138	88.8661	14.3722
3	03/02	6.2388	17.2679	8.4875	3.4001	16.2677	4.77293	6.1540	46.5987	12.1072
4	04/01	3.9339	11.9242	6.8348	2.6605	14.6089	4.24770	4.6732	57.7731	8.9771
5	05/02	3.9107	−0.4996	6.0362	10.0907	−0.9621	0.43764	5.1251	24.7135	2.1060
6	06/01	2.3813	4.2026	3.7167	3.9700	4.9608	3.79582	2.0483	17.5901	3.7462
7	07/01	0.8096	1.8277	2.6859	8.8962	1.2455	2.90534	1.4748	−19.5548	1.9469
8	08/01	−0.8444	4.2943	3.5904	0.6029	4.0917	−0.98798	−1.9689	−21.3192	1.8056
9	09/01	7.9378	12.2165	7.1177	2.1900	11.5797	1.54116	4.4171	−10.2306	8.4287
10	10/03	8.6066	6.0833	7.2786	3.0292	7.0678	2.25322	4.6167	−6.2692	5.6022
11	11/02	13.6189	8.0587	17.2973	11.0466	7.5563	8.58214	10.8964	7.1390	9.8846
12	12/01	8.9912	−7.6382	5.5608	13.8097	−5.8845	7.43355	8.2974	−0.0954	0.1376
Mean										6.3258981

2017

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/03	3.4250	−8.4376	1.73951	9.6032	−7.1264	3.35764	3.73341	12.7258	−2.3328
2	02/01	5.1585	−6.5170	1.90048	10.5896	−4.6838	5.31574	5.04964	−6.3986	−0.8560
3	03/01	0.9646	−9.6466	−0.68608	5.6463	−7.6968	4.34961	1.49446	2.4024	−3.8831
4	04/03	4.8472	−7.4433	2.19511	6.9089	−6.0870	6.90520	4.13523	−5.0966	−1.5257
5	05/01	5.2555	−4.7703	0.27012	4.7472	−2.3602	5.98408	3.23922	2.6611	−0.4617
6	06/01	4.4468	−1.9671	1.87186	3.1312	−0.1174	4.91891	2.52658	8.7123	0.7530
7	07/03	6.6846	5.1352	4.60933	8.2236	7.1092	5.79974	4.91047	8.6081	5.5486
8	08/01	7.1545	5.6497	3.66777	7.2812	7.1382	1.96897	4.60840	−2.5241	5.2055
9	09/01	11.2352	11.5111	7.10335	6.6879	10.7819	3.16588	7.51582	−14.5355	9.2106
10	10/02	13.2949	11.5855	6.99253	11.9028	10.5757	5.96048	9.21997	2.7655	10.3648
11	11/01	4.4129	−1.6694	0.26081	4.0779	−2.4124	−0.10224	2.41406	−0.6736	0.1099
12	12/01	3.4754	−3.4003	4.10364	12.2528	−3.5621	1.04077	5.21777	−0.4559	0.3952
Mean										1.8773428

2018

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	−2.0126	−2.0198	−1.1619	2.8139	−0.5522	−2.01207	−1.3909	−3.0583	−1.5585
2	02/01	−4.4280	3.3284	−0.5329	1.0993	4.9223	−4.76879	−3.1013	−3.8137	0.3903
3	03/01	3.4679	16.4257	8.7530	6.8584	15.0997	0.47361	4.5314	7.7423	10.5505
4	04/02	5.6514	16.7212	10.0850	14.1972	14.9197	−0.17724	8.4960	−0.8145	11.8413
5	05/01	7.0753	3.6356	9.3578	13.3821	2.0638	0.13592	8.6299	−7.0623	5.1465
6	06/01	4.9394	−4.6076	4.3728	5.6178	−4.8102	−3.12168	5.4467	−26.4328	−0.9678
7	07/02	6.2967	1.0911	−1.5716	1.0460	0.7774	−0.81743	3.1831	−20.7633	1.3421
8	08/01	3.1789	−9.6088	−3.9686	−1.5136	−9.1650	−6.46468	−0.3870	−14.0357	−5.7969
9	09/04	−5.1642	−16.0783	−14.8416	−11.3081	−14.4721	−7.64696	−8.5909	8.3807	−12.3852
10	10/01	−10.5199	−20.0203	−12.7862	−13.9683	−18.2936	−8.99819	−11.6697	10.6256	−15.5114
11	11/01	0.7965	−5.0291	3.4953	0.5407	−3.8969	0.63148	0.8262	13.0116	−1.8652
12	12/03	0.4486	−0.7976	3.1978	4.0017	0.5640	4.06158	1.7527	15.9072	0.9420
Mean										−0.6560326

2019

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	**01/02**	**14.7463**	20.1050	21.8037	22.5514	18.9169	14.3940	17.4632	8.4930	18.6857
2	02/01	1.6073	−1.7993	2.8720	7.0090	−0.9951	0.6420	4.5697	−9.0820	0.5407
3	03/01	0.3261	−6.4606	−1.0087	5.4769	−5.2840	0.6634	3.2701	−1.0657	−2.4620
4	04/01	3.2396	−4.3740	1.5343	6.0724	−3.4762	1.3364	4.9206	13.0707	−0.4490
5	05/01	−1.2440	−10.4825	−2.7418	−2.0465	−11.2138	−4.6617	−0.7939	38.5022	−6.0958
6	06/03	8.9338	1.9478	5.5960	12.6197	−1.5810	3.3759	9.0332	30.1200	4.7419
7	07/01	−1.3768	−12.9769	−4.5186	−1.0366	−12.1668	−4.6358	−1.5178	12.5226	−7.7422
8	08/01	3.8806	−2.0124	2.6335	5.3712	0.8451	4.3265	4.3579	6.0497	1.1125
9	09/03	7.0196	3.9311	7.7989	10.2382	6.4598	8.3393	8.3462	−2.2605	6.0319
10	10/01	8.3836	6.1473	8.9580	16.6298	6.7527	9.6613	10.9627	15.8383	8.3021
11	11/01	8.1162	−4.0231	6.6120	17.5180	−4.1342	5.0151	10.1550	8.8303	2.2237
12	12/02	−23.3853	−50.6019	−29.5760	−11.7555	−48.5921	−25.3600	−19.8796	−26.4071	−36.8615
Mean										−0.9976545

2020

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	−17.2711	−44.8229	−24.9950	−3.5981	−42.1996	−21.0806	−13.3032	−1.8441	−30.8412
2	02/03	−18.5455	−33.4829	−24.4459	−1.5603	−35.0776	−20.9909	−13.6760	13.2562	−25.1241
3	03/02	6.0889	−4.5949	1.4483	18.6579	−9.4019	−1.0652	8.0233	21.6612	0.3406
4	04/01	25.3064	29.2936	31.9343	42.4381	17.5152	20.5832	28.6800	68.6000	28.5032
5	05/01	16.8288	7.9461	23.2594	24.9348	9.9264	17.2252	17.8916	27.7860	13.5100
6	06/01	9.8486	−11.4994	6.5657	17.6745	−8.3207	9.7559	11.0604	24.7606	−0.5449
7	07/01	10.6054	−15.5080	13.1896	14.3758	−13.3512	10.3512	11.5086	17.4039	−2.1971
8	08/03	9.7708	−7.5136	10.7407	4.6809	−5.3039	6.5398	6.9228	−5.4017	−0.1046
9	09/01	5.6064	14.9474	16.3401	0.7694	13.9155	8.6440	4.3089	−7.0745	10.8928
10	10/01	11.6357	45.5622	28.5521	11.2730	43.2952	17.0781	12.3497	−0.5311	30.7619
11	11/02	17.4458	50.9228	31.9717	24.6385	46.6402	23.9315	19.0881	−1.2074	36.6048
12	12/01	9.5050	41.8682	17.7302	4.4356	35.3673	9.1791	7.6160	3.0860	25.8137
Mean										7.3012577

2021

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/04	13.0739	28.8827	16.6011	11.1154	23.1604	8.05119	13.0245	−3.0299	20.4584
2	02/01	11.7949	33.0073	9.0612	−0.1226	26.4386	6.56766	8.8914	7.4419	20.6125
3	03/01	10.8591	16.0292	6.5660	5.8180	15.2587	8.16446	9.7913	23.6540	12.6147
4	04/01	5.7345	7.1858	3.9300	11.9795	6.7819	5.40883	9.3704	3.0239	7.1665
5	05/03	4.1174	0.7762	1.0830	8.6178	2.3894	3.15679	5.9310	−7.1180	2.5767
6	06/01	0.8813	−8.4677	2.3144	14.3700	−5.8974	−2.31156	7.0155	−20.7218	−2.5521
7	07/01	−0.0358	5.1046	−0.7001	1.5579	6.8382	−3.62892	1.3919	−11.5984	2.5236
8	08/02	3.9211	19.6329	6.4087	8.1866	17.3475	1.20528	6.5239	−5.2286	12.3724
9	09/01	1.1998	18.2871	−2.1811	3.4308	14.5905	−3.10451	3.2777	−6.1970	9.5385
10	10/01	5.8719	16.5868	0.9376	6.4705	13.2745	6.95198	8.1555	7.7098	11.3861
11	11/01	−1.9477	15.9894	−7.9770	−7.2770	15.9902	5.01620	−2.4102	−0.3663	7.2177
12	12/01	−3.4958	33.8719	−8.1690	−17.9249	29.3687	1.13476	−7.7279	19.6871	14.4697
Mean										9.8653745

2022

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/03	−5.8097	39.0050	−8.3919	−14.6392	35.1174	0.8152	−7.6809	27.8701	17.1157
2	02/01	−10.7203	14.3496	−13.2277	−20.7286	10.3416	−13.8129	−13.9323	−3.4867	0.5829
3	03/01	−2.5368	27.0641	−6.1906	−11.8003	23.0641	−4.1736	−6.1308	−11.1289	10.9906
4	04/01	−12.7231	−14.3466	−18.3241	−21.9750	−15.8125	−18.4772	−17.4376	−35.9327	−16.0127
5	05/02	2.5885	2.6228	4.0429	4.3622	2.2595	−0.5257	3.4191	−19.7500	2.4184
6	06/01	−1.3051	−5.2927	2.5397	1.4597	−7.7598	−8.1749	0.2410	−16.8837	−3.6215
7	07/01	−4.6365	13.1779	−1.8967	−5.8206	7.3221	−9.7764	−4.8946	−6.8668	3.8299
8	08/01	0.5841	21.2289	−5.4066	−15.0830	14.3099	−6.3120	−7.5620	−0.1802	7.6816
9	09/01	7.3663	5.7343	1.0095	−4.0254	3.9575	6.5435	0.8108	25.6066	4.4330
10	10/03	17.1090	20.3218	12.4153	2.3761	19.3515	20.4021	9.2271	30.1316	16.8115
11	11/01	4.3163	−2.2495	7.8894	9.1784	1.1678	18.4392	6.4746	15.9123	3.4094
12	12/01	−7.9097	−16.5452	−8.4425	0.8175	−14.8383	−4.4085	−5.2490	−3.1959	−11.0891
Mean										3.0458044

2023

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/03	2.6796	3.7379	2.9670	20.3472	5.5387	8.8591	8.4039	14.0237	5.8007
2	02/01	−1.4020	−10.4468	−8.7360	10.3725	−7.4879	−1.6590	1.5337	1.0053	−5.4160
3	03/01	3.9582	−3.0206	−2.0772	21.6762	−2.6252	1.9168	8.9527	8.1986	1.4302
4	04/03	3.5480	−1.8165	7.3206	19.1343	0.3594	3.9981	9.7778	−2.5217	2.9177
5	05/01	3.1662	5.4054	6.8414	14.3033	5.3448	3.3762	7.0555	−11.6247	5.6369
6	06/01	5.6752	18.6168	7.5283	7.4469	18.0832	7.1140	6.8364	−6.8908	12.7575
7	07/03	−1.9666	6.3696	−5.2045	−0.2242	7.4408	−0.3655	−2.1824	−7.8415	2.2988
8	08/01	−4.3078	−3.9922	−11.0053	−2.6584	−1.5659	−3.8987	−4.5488	−10.8190	−4.4873
9	09/01	4.7595	−9.7860	−1.4088	4.8356	−6.1762	1.8212	2.3713	−2.5263	−3.7377
10	10/02	12.6574	−7.5187	10.3269	13.0418	−4.1932	8.3983	11.2523	13.3798	1.6435
11	11/01	15.2344	−3.6691	17.5682	21.0244	−2.7505	11.4974	17.2133	−5.3097	5.7637
12	12/01	7.7682	6.6445	13.2508	13.1273	4.3709	9.9353	12.4836	−2.6825	8.6136
Mean										2.7684716

2024

S/N	Date	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	Realized Return
1	01/02	1.01715	13.7180	3.45571	9.1104	10.4157	6.24252	8.15789	10.2313	9.70876
2	02/01	3.40152	12.9556	8.09014	5.6576	11.1229	8.08806	7.36867	23.0392	9.95689
3	03/01	−1.04055	4.9679	−0.67622	4.9939	4.1637	3.84277	4.57563	29.5565	3.98878
4	04/01	0.33400	−4.2155	−2.73800	10.7281	−2.7687	4.89417	6.64212	17.3573	−0.04640
5	05/01	3.89402	−1.6233	2.44344	6.6490	−2.2707	2.95198	6.20172	7.6816	1.36179
6	06/03	6.32284	−6.8531	1.32412	4.2095	−7.1793	3.28300	5.75395	5.0199	−1.34898
Mean										3.9368067

Table 8. Actual Realized 70-trading Days' Forward Return on the First Business Day of Each Month for Test Data

$$\left(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*, x_8^*\right) = \left(0.0778, 0.0104, 0.4012, 0.2661, 0.0011, 0.0393, 0.1169, 0.0872\right)$$

2024

S/N	*Date*	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	*Realized Return*
1	07/01	7.57061	−0.14023	8.3146	2.4023	−1.12721	3.91147	5.4804	12.1569	6.4159
2	08/01	9.06588	3.01454	10.8941	12.3457	0.41679	4.86167	10.4676	5.4848	10.2863
3	09/03	8.23396	2.39328	12.3677	14.1183	−0.50812	2.09292	10.0979	4.8013	11.0655
4	10/01	−0.24017	2.24171	1.3737	4.6684	0.79315	−4.32676	1.6199	−10.9915	0.8600
5	11/01	6.27932	1.94054	5.1663	10.3221	1.94792	4.99096	6.7165	1.3987	6.4337
6	12/02	−7.37508	−4.88677	−12.7294	−6.4338	−2.74236	2.64630	−6.6974	16.0407	−6.7270
Mean										4.7224040

2025

S/N	*Date*	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	*Realized Return*
1	01/02	−4.33401	−8.1189	−11.2925	−9.8345	−7.1581	−0.5860	−7.6008	36.3041	−5.3227
2	02/03	−4.38979	−1.7640	−3.1733	0.9308	−0.7229	6.8628	−0.8748	13.6225	−0.0310
3	03/03	−0.42100	−1.4335	2.3901	7.3758	1.5170	8.7077	3.2940	31.8068	6.3765
4	04/01	6.54477	−4.3240	11.8016	18.7488	−2.4446	10.5282	12.2461	20.1772	13.7904
5	05/01	8.21729	5.4047	11.9401	19.0311	7.7589	10.9065	14.0854	27.8110	15.0592
6	06/02	8.89641	8.2759	11.6256	12.6591	8.5958	10.1085	11.6758	36.9085	13.8013
7	07/01	5.12992	4.3134	6.0460	11.5183	4.0758	10.0717	8.7116	42.7913	11.0851
8	08/01	8.29168	4.0439	6.4561	11.8222	4.8173	11.1234	8.9795	39.0314	11.3193
9	09/02	5.59485	1.9567	3.7806	10.9950	1.9561	9.0173	7.2967	28.1492	8.5626
10	10/01	5.90997	3.5635	6.1661	4.4246	3.1629	7.2212	4.1746	27.5144	7.3228
11	11/03	4.13355	22.8892	8.7840	−5.3033	20.6296	13.0077	−0.3822	46.5248	7.2186
12	12/01	−1.64261	26.0025	2.1528	−3.2677	27.0294	4.6461	−2.5886	18.8840	1.6931
Mean										7.5729364

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