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A Comprehensive Study of Mean Estimators using Two Auxiliary Variables in Survey Sampling

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SUMMARY

This paper proposes a class of estimators for the finite population mean $\bar{Y}$ by utilizing auxiliary information from two correlated variables x and z under simple random sampling without replacement (SRSWOR). The bias and mean squared error (MSE) of the estimators are derived using first-order approximation. Theoretical comparisons establish conditions under which the proposed estimators outperform traditional estimators such as the unbiased, ratio, product, and difference estimators. An empirical study further confirms that the proposed estimators achieve lower MSE and higher relative efficiency, demonstrating improved precision when dual auxiliary information is incorporated in finite population sampling.

Keywords: Bias; Mean square error; Ratio estimator; Auxiliary variable; Study variable.

1. INTRODUCTION

In sample surveys, using auxiliary variables during the estimation stage significantly improved the precision of an estimate of the population mean. Several authors have focused on estimating the population mean $\bar{Y}$ of the study variable y using information on a single auxiliary variable x, and have proposed a large number of estimators as well as their properties in simple random sampling without (or with) replacement schemes, for example, Murthy (1967), Singh (1986, 2003), Sahoo & Jena (2024, 2024) and the references cited. In many survey circumstances with practical implications, adequate information on more than one auxiliary variable is available. The use of auxiliary information has been widely explored to improve the efficiency of estimators of the population mean. Olkin (1958) introduced the multivariate ratio estimator, which performs well when the auxiliary variables are strongly correlated with the study variable. Srivastava (1965, 1967, 1971) proposed generalized estimators that provide improved efficiency under certain correlation structures, although their performance depends heavily on the strength of the relationship between variables. Several authors, including Raj (1965), Rao and Mudholkar (1967), Singh (1966, 1969), Shukla (1966) Singh and Tailor (2005), Singh *et al.* (2009), Swain (2013), Sharma and Singh (2014, 2015) developed ratio–product and regression-type estimators that combine auxiliary information to reduce sampling variance. However, many of these estimators rely on single auxiliary variables or specific correlation assumptions. In practical surveys, information on more than one auxiliary variable is often available, and incorporating multiple auxiliary variables can significantly improve estimation accuracy. This motivates the development of estimators that effectively utilize two auxiliary variables for estimating the population mean.

In recent years, considerable attention has been given to the development of improved estimators of the population mean using auxiliary information. Several researchers have proposed efficient estimators based on auxiliary variables to reduce the mean squared error

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of estimators in sample surveys. For instance, Ahmad *et al.* (2023), Shabbir *et al.* (2023), and Oladugba and Babatunde (2024) developed improved estimation techniques using auxiliary information and multiple auxiliary variables, demonstrating significant gains in efficiency compared with traditional estimators.

The main contribution of this study is the development of a logarithmic-type estimator for estimating the finite population mean using two auxiliary variables under the framework of simple random sampling without replacement. Unlike existing estimators such as the classes proposed by Gupta and Shabbir (2008) and Singh and Nigam (2022), which are primarily based on ratio, product, or regression structures, the proposed estimator introduces a logarithmic transformation to better capture nonlinear relationships between the study and auxiliary variables. This transformation helps stabilize variability and reduces the effect of large discrepancies between sample and population means. Consequently, the proposed estimator provides a more flexible and efficient estimation framework and includes several traditional estimators as special cases.

This study proposes an extensive variety of estimator for the finite population mean $\bar{Y}$ of the variable under study y, by utilising data on two auxiliary variables, x and z. The proposed estimators are explored up to the first order of approximation in SRSWOR. Numerical examples support the current study.

2. SOME EXISTING ESTIMATORS OF SRS

Consider a finite population $\Omega=\{\Omega_1,\Omega_2,\ldots,\Omega_N\}$ of N units. Let y and (x,z) be study variable and auxilliary variables respectively. Let y_i and (x_i,z_i) be the values of study variable y and auxiliary varibles (x,z) on the i^{th} unit Ω_i of the population Ω. Suppose a sample of size n is drawn by using SRSWOR scheme from the population Ω for estimating the population mean $\bar{Y}$ of the study variable y. Let $\bar{y}=\frac{1}{n}\sum_{i=1}^{n}y_i$ and $\left(\bar{x}=\frac{1}{n}\sum_{i=1}^{n}x_i,\bar{z}=\frac{1}{n}\sum_{i=1}^{n}z_i\right)$ be the unbised estimators of the population means $\bar{Y}=\frac{1}{N}\sum_{i=1}^{N}Y_i$ and $\left(\bar{Z}=\frac{1}{N}\sum_{i=1}^{N}X_i,\bar{Z}=\frac{1}{N}\sum_{i=1}^{N}Z_i\right)$ respectively.

It is assumed that the population means $(\bar{X},\bar{Z})$ of (x,z) are known. Further we denoted

$C_y=\frac{S_y}{\bar{Y}}-$ the population coefficient of variation of y,

$C_x=\frac{S_x}{\bar{X}}-$ the population coefficient of variation of x,

$C_z=\frac{S_z}{\bar{Z}}-$ the population coefficient of variation of z,

$\rho_{yx}=\frac{S_{yx}}{S_yS_x}-$ the population correlation coefficient between y and x,

$\rho_{yz}=\frac{S_{yz}}{S_yS_z}-$ the population correlation coefficient between y and z,

$\rho_{xz}=\frac{S_{xz}}{S_xS_z}-$ the population correlation coefficient between x and z,

$S_{xy}=\frac{1}{N-1}\sum_{i=1}^{N}(y_i-\bar{Y})(x_i-\bar{X})-$ the population covariance between y and x,

$S_{yz}=\frac{1}{N-1}\sum_{i=1}^{N}(y_i-\bar{Y})(z_i-\bar{Z})-$ the population covariance between y and z,

$S_{xz}=\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{X})(z_i-\bar{Z})-$ the population covariance between x and z,

$S_y^2=\frac{1}{N-1}\sum_{i=1}^{N}(y_i-\bar{Y})^2-$ the population mean square of y,

$S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{X}\right)^2 -$ the population mean square of x,

$S_z^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(z_i - \bar{Z}\right)^2 -$ the population mean square of z,

$K_{yx} = \frac{\rho_{yx}C_y}{C_x},\ K_{yz} = \frac{\rho_{yz}C_y}{C_z},\ K_{zx} = \frac{\rho_{zx}C_z}{C_x},\ K_{xz} = \frac{\rho_{zx}C_x}{C_z},$

and $f = \frac{n}{N}$ is the sampling fraction.

The usual unbiased estimator for population mean $\bar{Y}$ is given by

$$\hat{\hat{Y}}_0 = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \tag{1}$$

The variance/mean squared error (MSE) under $SRSWOR$ is given by

$$Var\left(\hat{\hat{Y}}_0\right) = MSE\left(\hat{\hat{Y}}_0\right) = \left(\frac{1-f}{n}\right)\bar{Y}^2 C_y^2 = \left(\frac{1-f}{n}\right)S_y^2 \tag{2}$$

The usual ratio and product estimators for population mean $\bar{Y}$ are respectively defined by

$$\hat{\hat{Y}}_1 = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right) \tag{3}$$

and

$$\hat{\hat{Y}}_2 = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right) \tag{4}$$

To the first degree of approximation, the $MSEs$ of the estimators $\hat{\hat{Y}}_1$ and $\hat{\hat{Y}}_2$ are respectively given by

$$MSE\left(\hat{\hat{Y}}_1\right) = \left(\frac{1-f}{n}\right)\bar{Y}^2\left[C_y^2 + C_x^2\left(1 - 2K_{yx}\right)\right] \tag{5}$$

$$MSE\left(\hat{\hat{Y}}_2\right) = \left(\frac{1-f}{n}\right)\bar{Y}^2\left[C_y^2 + C_x^2\left(1 + 2K_{yx}\right)\right] \tag{6}$$

The generalized version of the estimators $\hat{\hat{Y}}_0$, $\hat{\hat{Y}}_1$ and $\hat{\hat{Y}}_2$ due to Srivastava [7] is given by

$$\hat{\hat{Y}}_3 = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha_1}, \tag{7}$$

where α_1 is a suitably choosen constant.

The $MSE\left(\hat{\hat{Y}}_3\right)$ is in the first order of approximation given by

$$MSE\left(\hat{\hat{Y}}_3\right) = \left(\frac{1-f}{n}\right)\bar{Y}^2\left[C_y^2 + \alpha_1 C_x^2\left(\alpha_1 + 2K_{yx}\right)\right], \tag{8}$$

which is minimum when

$$\alpha_1 = -K_{yx}\alpha_{1(opt)}\text{, say} \tag{9}$$

Substitution of the equation (9) in the equation (8) yields the minimum MSE of $\hat{\hat{Y}}_3$ as

$$MSE\left(\hat{\hat{Y}}_3\right)_{min.} = \left(\frac{1-f}{n}\right)\bar{Y}^2\left[C_y^2 - K_{yx}^2 C_x^2\right]$$

$$= \left(\frac{1-f}{n}\right)\bar{Y}^2 C_y^2\left[1-\rho_{yx}^2\right] = \left(\frac{1-f}{n}\right)S_y^2\left[1-\rho_{yx}^2\right] \tag{10}$$

The traditional difference estimator for $\bar{Y}$ is defined by

$$\hat{\hat{Y}}_4 = \bar{y} + d_0\left(\bar{X} - \bar{x}\right), \tag{11}$$

where d_0 is a suitable chosen constant to be determined such that the $MSE\left(\hat{\hat{Y}}_4\right)$ is minimum.

$$MSE\left(\hat{\hat{Y}}_4\right)_{min.} = \left(\frac{1-f}{n}\right)\bar{Y}^2 C_y^2\left[1-\rho_{yx}^2\right] = \left(\frac{1-f}{n}\right)S_y^2\left[1-\rho_{yx}^2\right] \tag{12}$$

Upadhaya *et al.* (1985) suggested a class of estimators for population mean $\bar{Y}$ as

$$\hat{\hat{Y}}_5 = w_0\bar{y} + w_1\bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha_1}, \tag{13}$$

where w_0 and w_1 are suitably chosen weights whose sum need not be 'unity' and α_1 is a designed parameter. To the first order of approximation, $MSE\left(\hat{\hat{Y}}_5\right)$ is given by

$$MSE\left(\hat{\hat{Y}}_5\right)_{min.} = \bar{Y}^2\left[1 + w_0^2 A_{0(srs)} + w_1^2 A_{1(srs)} + 2w_0 w_1 A_{3(srs)} - 2w_0 - 2w_1 A_{6(srs)}\right], \tag{14}$$

where

$$A_{0(srs)}=\left[1+\left(\frac{1-f}{n}\right)C_y^2\right],$$

$$A_{1(srs)}=\left[1+\left(\frac{1-f}{n}\right)\left\{C_y^2+4\alpha_1\rho_{yx}C_yC_x+\alpha_1\left(2\alpha_1-1\right)C_x^2\right\}\right],$$

$$A_{3(srs)}=\left[1+\left(\frac{1-f}{n}\right)\left\{C_y^2+2\alpha_1\rho_{yx}C_yC_x+\frac{\alpha_1\left(\alpha_1-1\right)}{2}C_x^2\right\}\right],$$

$$A_{6(srs)}=\left[1+\left(\frac{1-f}{n}\right)\left\{\alpha_1\rho_{yx}C_yC_x+\frac{\alpha_1\left(\alpha_1-1\right)}{2}C_x^2\right\}\right]$$

The best values of $\left(w_0,w_1\right)$ for which the $MSE\left(\hat{\hat{Y}}_5\right)$ is minimized and given by

$$w_0=\frac{\Delta_0^*}{\Delta^*},\ w_1=\frac{\Delta_1^*}{\Delta^*}, \tag{15}$$

where

$$\Delta^*=\begin{vmatrix}A_{0(srs)} & A_{3(srs)}\\ A_{3(srs)} & A_{1(srs)}\end{vmatrix}=\left(A_{0(srs)}A_{1(srs)}-A_{3(srs)}^2\right)$$

$$\Delta_0^*=\begin{vmatrix}1 & A_{3(srs)}\\ A_{6(srs)} & A_{1(srs)}\end{vmatrix}=\left(A_{1(srs)}-A_{3(srs)}A_{6(srs)}\right)$$

$$\Delta_1^*=\begin{vmatrix}A_{0(srs)} & A_{1(srs)}\\ A_{3(srs)} & A_{6(srs)}\end{vmatrix}=\left(A_{0(srs)}A_{6(srs)}-A_{3(srs)}A_{1(srs)}\right)$$

Thus

$$MSE\left(\hat{\hat{Y}}_5\right)_{min.}=\bar{Y}^2\left[1-\frac{\left\{A_{1(srs)}-2A_{3(srs)}A_{6(srs)}+A_{0(srs)}A_{6(srs)}^2\right\}}{\left(A_{0(srs)}A_{1(srs)}-A_{3(srs)}^2\right)}\right] \tag{16}$$

If we set $w_0+w_1=1\Rightarrow w_1=1-w_0$ in the equation (13), we get an estimator for population mean $\bar{Y}$ of y as

$$\hat{\hat{Y}}_5^{\ *}=w_0\bar{y}+\left(1-w_0\right)\bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)^{\alpha_1} \tag{17}$$

which includes the estimators due to Srivastava (1967), Ray *et al.* (1979) and Vos(1980).

Putting $w_1=1-w_0$ in the equation (14), we get the $MSE\left(\hat{\hat{Y}}_5^{\ *}\right)$ as

$$MSE\left(\hat{\hat{Y}}_5^{\ *}\right)=\bar{Y}^2\left[1+A_{1(srs)}-2A_{6(srs)}+w_0^2\left(A_{0(srs)}+A_{1(srs)}-2A_{3(srs)}\right)-2w_0\left(1+A_{1(srs)}-A_{3(srs)}-A_{6(srs)}\right)\right] \tag{18}$$

which is minimized for

$$w_0=\frac{\left(1+A_{1(srs)}-A_{3(srs)}-A_{6(srs)}\right)}{\left(A_{0(srs)}+A_{1(srs)}-2A_{3(srs)}\right)}=\left(1+\frac{K_{yx}}{\alpha_1}\right)=w_{0(opt)},$$

say (19)

Thus

$$MSE\left(\hat{\hat{Y}}_5^{\ *}\right)_{min.}=\bar{Y}^2\left[1+A_{1(srs)}-2A_{6(srs)}-\frac{\left(1+A_{1(srs)}-A_{3(srs)}-A_{6(srs)}\right)^2}{\left(A_{0(srs)}+A_{1(srs)}-2A_{3(srs)}\right)}\right]$$

$$=\left(\frac{1-f}{n}\right)\bar{Y}^2C_y^2\left[1-\rho_{yx}^2\right]=\left(\frac{1-f}{n}\right)S_y^2\left[1-\rho_{yx}^2\right] \tag{20}$$

Rao (1991) suggested difference-type estimator for population mean $\bar{Y}$ as

$$\hat{\hat{Y}}_6=\alpha_1\bar{y}+\alpha_2\left(\bar{X}-\bar{x}\right), \tag{21}$$

where α_1 and α_2 are constants to be determined such that MSE of $\hat{\hat{Y}}_6$ is minimum.

The bias and MSE of $\hat{\hat{Y}}_6$ are respectively given by

$$B\left(\hat{\hat{Y}}_6\right)=\bar{Y}\left(\alpha_1-1\right) \tag{22}$$

and

$$MSE\left(\hat{\hat{Y}}_6\right)=\bar{Y}^2\left[1+\alpha_1^2\left\{1+\left(\frac{1-f}{n}\right)C_y^2\right\}+\alpha_2^2\left(\frac{1-f}{n}\right)\frac{C_x^2}{R^2}-2\alpha_1\alpha_2\left(\frac{1-f}{n}\right)\frac{K_{yx}C_x^2}{R}-2\alpha_1\right], \tag{23}$$

where $R=\frac{\bar{Y}}{\bar{X}}$

The $MSE\left(\hat{\hat{Y}}_6\right)$ at the equation (23) is minimized for

$$\alpha_1 = \left\{1+\left(\frac{1-f}{n}\right)\left(C_y^2 - K_{yx}^2 C_x^2\right)\right\}^{-1} = \alpha_{1(opt)},\ \text{say}$$

$$\alpha_2 = -RK_{yx}\left\{1+\left(\frac{1-f}{n}\right)\left(C_y^2 - K_{yx}^2 C_x^2\right)\right\}^{-1} = \alpha_{2(opt)},$$

say

Hence

$$MSE\left(\hat{\hat{Y}}_6\right)_{min.} = \frac{\left(\frac{1-f}{n}\right)\bar{Y}^2\left(C_y^2 - K_{yx}^2 C_x^2\right)}{1+\left(\frac{1-f}{n}\right)\left(C_y^2 - K_{yx}^2 C_x^2\right)} \tag{24}$$

Gupta and Shabbir (2008) proposed the following class of estimators for population mean $\bar{Y}$ as

$$\hat{\hat{Y}}_7 = \left[\alpha_3 \bar{y} + \alpha_4\left(\bar{X}-\bar{x}\right)\right]\left(\frac{\bar{X}}{\bar{x}}\right), \tag{25}$$

where (α_3, α_4) are suitably chosen constants such that $MSE\left(\hat{\hat{Y}}_7\right)$ is minimum.

To the first order of approximation, the bias and MSE of $\hat{\hat{Y}}_7$ are respectively given by

$$B\left(\hat{\hat{Y}}_7\right) = \bar{Y}\left[\alpha_3\left\{\left(\frac{1-f}{n}\right)C_x^2\left(1-K_{yx}\right)\right\} + \frac{\alpha_4}{R}C_x^2 - 1\right] \tag{26}$$

$$MSE\left(\hat{\hat{Y}}_7\right) = \bar{Y}^2\left[1+\alpha_3^2\left\{1+\left(\frac{1-f}{n}\right)\left(C_y^2 + C_x^2\left(3-4K_{yx}\right)\right)\right\} + \alpha_4^2\left(\frac{1-f}{n}\right)\frac{C_x^2}{R^2} + \frac{2\alpha_3\alpha_4}{R}\left(\frac{1-f}{n}\right)C_x^2\left(2-K_{yx}\right) - 2\alpha_3\left\{1+\left(\frac{1-f}{n}\right)C_x^2\left(1-K_{yx}\right) - 2\alpha_3\left(\frac{1-f}{n}\right)\frac{C_x^2}{R}\right\}\right] \tag{27}$$

The $MSE\left(\hat{\hat{Y}}_7\right)$ at the equation (27) is minimum when

$$\alpha_3 = \frac{\left(a_2 a_4 - a_3 a_5\right)}{\left(a_1 a_2 - a_3^2\right)} = \alpha_{3(opt)},\ \text{say}$$

$$\alpha_4 = \frac{\left(a_2 a_5 - a_3 a_4\right)}{\left(a_1 a_2 - a_3^2\right)} = \alpha_{4(opt)},\ \text{say}$$

where

$$a_1 = \left[1+\left(\frac{1-f}{n}\right)\left(C_y^2 + C_x^2\left(3-4K_{yx}\right)\right)\right]$$

$$a_2 = \left(\frac{1-f}{n}\right)\frac{C_x^2}{R^2}$$

$$a_3 = \left(\frac{1-f}{n}\right)C_x^2\left(2-K_{yx}\right)$$

$$a_4 = \left[\left\{1+\left(\frac{1-f}{n}\right)C_x^2\left(1-K_{yx}\right)\right\}\right]$$

$$a_5 = \left(\frac{1-f}{n}\right)\frac{C_x^2}{R}$$

Thus the minimum MSE of $\hat{\hat{Y}}_7$ is given by

$$MSE\left(\hat{\hat{Y}}_7\right)_{min.} = \bar{Y}^2\left\{1-\frac{\left(a_1 a_4^2 - 2a_3 a_4 a_5 + a_1 a_5^2\right)}{\left(a_1 a_2 - a_3^2\right)}\right\} \tag{28}$$

The traditional difference estimator for the population mean $\bar{Y}$ using two auxiliary variables x and z is defined by

$$\hat{\hat{Y}}_8 = \left[\bar{y} + k_1\left(\bar{X}-\bar{x}\right) + k_2\left(\bar{Z}-\bar{z}\right)\right], \tag{29}$$

where k_1 and k_2 are constants to be determined such that $MSE\left(\hat{\hat{Y}}_8\right)$ is minimum.

It is obvious from the equation (29) that the difference estimator $\hat{\hat{Y}}_8$ is unbiased for population mean $\bar{Y}$.

The variance/ MSE of the estimator $\hat{\hat{Y}}_8$ is given by

$$Var\left(\hat{\hat{Y}}_8\right) = MSE\left(\hat{\hat{Y}}_8\right) = \left(\frac{1-f}{n}\right)\bar{Y}^2\left[C_y^2 + k_1^2\frac{C_x^2}{R^2} + k_2^2\frac{C_z^2}{R^2} + 2k_1k_2\frac{K_{xz}C_x^2}{RR^*} - 2k_1\frac{K_{yx}C_x^2}{R} - 2k_2\frac{K_{yz}C_z^2}{R^*}\right], \tag{30}$$

where $R^* = \dfrac{\overline{Y}}{\overline{Z}}$

The $MSE\left(\hat{\hat{\overline{Y}}}_8\right)$ at the equation (30) is minimized for

$$k_1 = \frac{RC_y\left(\rho_{yx} - \rho_{yz}\rho_{xz}\right)}{C_y\left(1-\rho_{xz}^2\right)} = k_{1(opt)} \text{, say}$$

$$k_2 = \frac{RC_y\left(\rho_{yz} - \rho_{yx}\rho_{xz}\right)}{C_z\left(1-\rho_{xz}^2\right)} = k_{2(opt)} \text{, say}$$

Thus the minimum MSE of $\hat{\hat{\overline{Y}}}_8$ is given by

$$MSE\left(\hat{\hat{\overline{Y}}}_8\right)_{min.} = \left(\frac{1-f}{n}\right)\overline{Y}^2 C_y^2\left[1-R_{y.xz}^2\right] = \left(\frac{1-f}{n}\right)S_y^2\left[1-R_{y.xz}^2\right], \tag{31}$$

where $R_{y.xz}^2 = \dfrac{\rho_{yx}^2 + \rho_{yz}^2 - 2\rho_{yx}\rho_{yz}\rho_{xz}}{1-\rho_{xz}^2}$ is the multiple correlation coefficient.

Singh and Nigam (2022) proposed the following general class of estimators for the population mean $\overline{Y}$ using two auxiliary variables x and z is defined by

$$\hat{\hat{\overline{Y}}}_9 = w_0\overline{y} + w_1\overline{y}\left(\frac{\overline{x}}{\overline{\overline{X}}}\right)^{\alpha_1} + w_2\overline{y}\left(\frac{\overline{z}}{\overline{\overline{Z}}}\right)^{\alpha_2}, \tag{32}$$

where (w_0, w_1, w_2) are weights whose sum need not be 'unity' and (α_1, α_2) are designed parameters.

The bias of $\hat{\hat{\overline{Y}}}_8$ is given by

$$B\left(\hat{\hat{\overline{Y}}}_9\right) = \overline{Y}\left[w_0 + w_1 A_{6(srs)} + w_2 A_{7(srs)} - 1\right], \tag{33}$$

where

$$A_{6(srs)} = \left[1 + \left(\frac{1-f}{n}\right)\frac{\alpha_1}{2}\left(\alpha_1 + 2K_{yx} - 1\right)C_x^2\right]$$

$$A_{7(srs)} = \left[1 + \left(\frac{1-f}{n}\right)\frac{\alpha_1}{2}\left(\alpha_1 + 2K_{yz} - 1\right)C_z^2\right]$$

The MSE of $\hat{\hat{\overline{Y}}}_9$ to the first degree of approximation is thus

$$MSE\left(\hat{\hat{\overline{Y}}}_9\right) = \overline{Y}^2\left[1 + w_0^2 A_{0(srs)} + w_1^2 A_{1(srs)} + w_2^2 A_{2(srs)} + 2w_0w_1A_{3(srs)} + 2w_0w_2A_{4(srs)} + 2w_1w_2A_{5(srs)} - 2w_0 - 2w_1A_{6(srs)} - 2w_2A_{7(srs)}\right], \tag{34}$$

where

$$A_{2(srs)} = \left[1 + \left(\frac{1-f}{n}\right)\left\{C_y^2 + 4\alpha_2\rho_{yz}C_yC_z + \alpha_2\left(2\alpha_2 - 1\right)C_z^2\right\}\right]$$

$$A_{4(srs)} = \left[1 + \left(\frac{1-f}{n}\right)\left\{C_y^2 + 2\alpha_2\rho_{yz}C_yC_z + \frac{\alpha_2\left(\alpha_2 - 1\right)}{2}C_z^2\right\}\right]$$

$$A_{5(srs)} = \left[1 + \left(\frac{1-f}{n}\right)\left\{C_y^2 + 2\alpha_1\rho_{yx}C_yC_x + 2\alpha_2\rho_{yz}C_yC_z + \alpha_1\alpha_2\rho_{xz}C_zC_x + \frac{\alpha_1\left(\alpha_1 - 1\right)}{2}C_x^2 + \frac{\alpha_2\left(\alpha_2 - 1\right)}{2}C_z^2\right\}\right] \tag{35}$$

$\left(A_{0(srs)}, A_{1(srs)}, A_{3(srs)}, A_{6(srs)}\right)$ and $A_{7(srs)}$ are same as defined earlier and minimization of $MSE\left(\hat{\hat{\overline{Y}}}_9\right)$ at the equation (35) with respect to (w_0, w_1, w_2) yields

$$\begin{bmatrix} A_{0(srs)} & A_{3(srs)} & A_{4(srs)} \\ A_{3(srs)} & A_{1(srs)} & A_{5(srs)} \\ A_{4(srs)} & A_{5(srs)} & A_{2(srs)} \end{bmatrix}\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ A_{6(srs)} \\ A_{7(srs)} \end{bmatrix} \tag{36}$$

After simplification of equation (36), we get the optimum values of (w_0, w_1, w_2) respectively as

$$w_{00} = \frac{\Delta_0^{**}}{\Delta^{**}},\ w_{10} = \frac{\Delta_1^{**}}{\Delta^{**}},\ w_{20} = \frac{\Delta_2^{**}}{\Delta^{**}}, \tag{37}$$

where

$$\Delta^{**} = \begin{vmatrix} A_{0(srs)} & A_{3(srs)} & A_{4(srs)} \\ A_{3(srs)} & A_{1(srs)} & A_{5(srs)} \\ A_{4(srs)} & A_{5(srs)} & A_{2(srs)} \end{vmatrix}$$

$$= A_{0(srs)}\left(A_{1(srs)}A_{2(srs)} - A_{5(srs)}^2\right) - A_{3(srs)}\left(A_{2(srs)}A_{3(srs)} - A_{4(srs)}A_{5(srs)}\right) + A_{4(srs)}\left(A_{3(srs)}A_{5(srs)} - A_{1(srs)}A_{4(srs)}\right)$$

$$\Delta_0^{**} = \begin{bmatrix} 1 & A_{3(srs)} & A_{4(srs)} \\ A_{6(srs)} & A_{1(srs)} & A_{5(srs)} \\ A_{7(srs)} & A_{5(srs)} & A_{2(srs)} \end{bmatrix}$$

$$= \left(A_{1(srs)}A_{2(srs)} - A_{5(srs)}^2\right) - A_{3(srs)}\left(A_{2(srs)}A_{6(srs)} - A_{5(srs)}A_{7(srs)}\right) + A_{4(srs)}\left(A_{5(srs)}A_{6(srs)} - A_{1(srs)}A_{7(srs)}\right)$$

$$\Delta_1^{**} = \begin{bmatrix} A_{0(srs)} & 1 & A_{4(srs)} \\ A_{3(srs)} & A_{6(srs)} & A_{5(srs)} \\ A_{4(srs)} & A_{7(srs)} & A_{2(srs)} \end{bmatrix}$$

$$= A_{0(srs)}\left(A_{2(srs)}A_{6(srs)} - A_{5(srs)}A_{7(srs)}\right) - \left(A_{2(srs)}A_{3(srs)} - A_{4(srs)}A_{5(srs)}\right) + A_{4(srs)}\left(A_{3(srs)}A_{7(srs)} - A_{4(srs)}A_{6(srs)}\right)$$

$$\Delta_2^{**} = \begin{bmatrix} A_{0(srs)} & A_{3(srs)} & 1 \\ A_{3(srs)} & A_{1(srs)} & A_{6(srs)} \\ A_{4(srs)} & A_{5(srs)} & A_{7(srs)} \end{bmatrix}$$

$$= A_{0(srs)}\left(A_{1(srs)}A_{7(srs)} - A_{5(srs)}A_{6(srs)}\right) - A_{3(srs)}\left(A_{3(srs)}A_{7(srs)} - A_{4(srs)}A_{6(srs)}\right) + \left(A_{3(srs)}A_{5(srs)} - A_{4(srs)}A_{6(srs)}\right)$$

Substitution of equation (37) in the equation (35) yields the minimum MSE of $\hat{\hat{\bar{Y}}}_9$

$$MSE\left(\hat{\hat{\bar{Y}}}_9\right)_{min.} = \bar{Y}^2\left[1 - \frac{\Delta_0^{**}}{\Delta^{**}} - \frac{A_{6(srs)}\Delta_1^{**}}{\Delta^{**}} - \frac{A_{7(srs)}\Delta_2^{**}}{\Delta^{**}}\right] \tag{38}$$

3. SUGGESTED AN EXTENSIVE VARIETY OF ESTIMATOR IN SRS

Motivated by Singh and Nigam (2022), we propose an extensive variety of estimator based on two auxiliary variables x and z for population mean $\bar{Y}$ of y as

$$\hat{\hat{\bar{Y}}}_{10} = w_4\bar{y} + w_5\bar{y}\log\left(\frac{\bar{x}}{\bar{X}}\right) + w_6\bar{y}\log\left(\frac{\bar{z}}{\bar{Z}}\right), \tag{39}$$

where (w_4, w_5, w_6) are the weights whose sum need not be 'unity'.

In deriving the bias and mean squared error (MSE) of $\hat{\hat{\bar{Y}}}_{10}$, a first-order large sample approximation is employed. This approximation is based on the assumption that the sampling errors associated with the sample means are relatively small compared to their corresponding population means

$\bar{y} = \bar{Y}(1+e_0)$, $\bar{x} = \bar{X}(1+e_1)$ and $\bar{z} = \bar{Z}(1+e_2)$, where e_0, e_1 and e_2 random error terms such that

$$E(e_0) = E(e_1) = E(e_2) = 0, \; E(e_0^2) = \left(\frac{1-f}{n}\right)C_y^2,$$

$$E(e_1^2) = \left(\frac{1-f}{n}\right)C_x^2, \; E(e_2^2) = \left(\frac{1-f}{n}\right)C_z^2,$$

$$E(e_0e_1) = \left(\frac{1-f}{n}\right)\rho_{yx}C_yC_x = \left(\frac{1-f}{n}\right)K_{yx}C_x^2,$$

$$E(e_0e_2) = \left(\frac{1-f}{n}\right)\rho_{yz}C_yC_z = \left(\frac{1-f}{n}\right)K_{yz}C_z^2,$$

$$E(e_1e_2) = \left(\frac{1-f}{n}\right)\rho_{xz}C_xC_z = \left(\frac{1-f}{n}\right)K_{xz}C_x^2 = \left(\frac{1-f}{n}\right)K_{zx}C_z^2.$$

Under the framework of simple random sampling without replacement (SRSWOR), these error terms are assumed to be sufficiently small so that higher-order can be ignored. Consequently, only terms up to the second orders are retained while deriving expressions for bias and MSE.

This approximation is commonly used in survey sampling literature because it simplifies analytical derivations while providing reasonably accurate results when the sample size is moderate to large and the sampling fraction is not extremely high. However, when the sample size is very small, ignoring higher-order terms may slightly affect the accuracy of the approximation. Therefore, the derived bias and MSE expressions should be interpreted as approximate measures of estimator performance under the large sample assumption.

Theorem 3.1

The bias of the proposed estimator $\hat{\hat{\bar{Y}}}_{10}$ of population mean $\bar{Y}$ is given by

$$Bias\left(\hat{\hat{Y}}_{10}\right)=\bar{Y}\left\{w_4-w_5\left(\frac{1-f}{n}\right)\left(\frac{C_x^2}{2}-K_{yx}C_x^2\right)-w_6\left(\frac{1-f}{n}\right)\left(\frac{C_z^2}{2}-K_{yz}C_z^2\right)-1\right\}$$

$$=\bar{Y}\left\{w_4+w_5B_{3(srs)}+w_6B_{4(srs)}-1\right\},$$

where

$$B_{3(srs)}=\left(\frac{1-f}{n}\right)\left(\frac{-C_x^2}{2}+K_{yx}C_x^2\right)$$

$$B_{4(srs)}=\left(\frac{1-f}{n}\right)\left(\frac{-C_z^2}{2}+K_{yz}C_z^2\right).$$

Proof.

Using the Taylor expansion

$$\log(1+e)\approx e-\frac{e^2}{2}$$

We obtain

$$\log(1+e_1)\approx e_1-\frac{e_1^2}{2}$$

$$\log(1+e_2)\approx e_2-\frac{e_2^2}{2}$$

So,

$$\hat{\hat{Y}}_{10}=w_4\bar{y}+w_5\bar{y}\log\left(\frac{\bar{x}}{\bar{X}}\right)+w_6\bar{y}\log\left(\frac{\bar{z}}{\bar{Z}}\right)$$

$$=w_4\bar{Y}(1+e_0)+w_5\bar{Y}(1+e_0)\log\left(\frac{\bar{X}(1+e_1)}{\bar{X}}\right)+w_6\bar{Y}(1+e_0)\log\left(\frac{\bar{Z}(1+e_2)}{\bar{Z}}\right)$$

$$=w_4\bar{Y}(1+e_0)+w_5\bar{Y}(1+e_0)\log(1+e_1)+w_6\bar{Y}(1+e_0)\log(1+e_2) \tag{40}$$

Substituting the above logarithm expansion into equation (40), we get

$$\hat{\hat{Y}}_{10}=\bar{Y}\left\{w_4(1+e_0)+w_5(1+e_0)\left(e_1-\frac{e_1^2}{2}\right)+w_{6(1+e_0)}\left(e_2-\frac{e_2^2}{2}\right)\right\}$$

Multiplying out and ignoring terms of e's having power greater than two, we have

$$\hat{\hat{Y}}_{10}=\bar{Y}\left\{w_4(1+e_0)+w_5\left(e_1-\frac{e_1^2}{2}+e_0e_1\right)+w_6\left(e_2-\frac{e_2^2}{2}+e_0e_2\right)\right\}$$

$$\Rightarrow\left(\hat{\hat{Y}}_{10}-\bar{Y}\right)=\bar{Y}\left\{w_4(1+e_0)+w_5\left(e_1-\frac{e_1^2}{2}+e_0e_1\right)+w_6\left(e_2-\frac{e_2^2}{2}+e_0e_2\right)-1\right\} \tag{41}$$

Taking expectation of both sides of (41) we get the bias of $\hat{\hat{Y}}_{10}$ to the first order of approximation as

$$Bias\left(\hat{\hat{Y}}_{10}\right)=E\left(\hat{\hat{Y}}_{10}-\bar{Y}\right)$$

$$=\bar{Y}\left\{w_4-w_5\left(\frac{1-f}{n}\right)\left(\frac{C_x^2}{2}-K_{yx}C_x^2\right)-w_6\left(\frac{1-f}{n}\right)\left(\frac{C_z^2}{2}-K_{yz}C_z^2\right)-1\right\}$$

$$=\bar{Y}\left\{w_4+w_5B_{3(srs)}+w_6B_{4(srs)}-1\right\}. \tag{42}$$

Theorem 3.2

The minimum mean squared error of the proposed estimator $\hat{\hat{Y}}_{10}$, to the first order approximation is

$$MSE\left(\hat{\hat{Y}}_{10}\right)_{min.}=\bar{Y}^2\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right].$$

Proof.

From equation (41), we have

$$\left(\hat{\hat{Y}}_{10}-\bar{Y}\right)=\bar{Y}\left\{w_4\left(1+e_0\right)+w_5\left(e_1-\frac{e_1^2}{2}+e_0e_1\right)+ w_6\left(e_2-\frac{e_2^2}{2}+e_0e_2\right)-1\right\} \tag{43}$$

Squaring both sides of (43) and ignoring terms of e 's having power greater than two, we have

$$\left(\hat{\hat{Y}}_{10}-\bar{Y}\right)^2=\bar{Y}^2\Bigg[1+w_4^2\left(1+e_0^2+e_0\right)+w_5^2e_1^2+w_6^2e_1^2+ 2w_4w_5\left(e_1-\frac{e_1^2}{2}+e_0e_1\right)+2w_4w_6\left(e_2-\frac{e_2^2}{2}+e_0e_2\right)+ 2w_5w_6e_0e_2-2w_4\left(1+e_0\right)-2w_5\left(e_1-\frac{e_1^2}{2}\right)-2w_6\left(e_2-\frac{e_2^2}{2}\right)\Bigg] \tag{44}$$

Taking expectation of both sides of (44), we get the MSE of $\hat{\hat{Y}}_{10}$ to the first order approximation as

$$E\left(\hat{\hat{Y}}_{10}\right)=\bar{Y}^2\Big[1+w_4^2B_{0(srs)}+w_5^2B_{1(srs)}+w_6^2B_{2(srs)}+ 2w_4w_5B_{3(srs)}+2w_4w_6B_{4(srs)}+2w_5w_6B_{5(srs)}- 2w_0-2w_5B_{6(srs)}-2w_6B_{7(srs)}\Big], \tag{45}$$

where

$$B_{0(srs)}=1+\left(\frac{1-f}{n}\right)C_y^2,$$

$$B_{1(srs)}=\left(\frac{1-f}{n}\right)C_x^2,$$

$$B_{2(srs)}=\left(\frac{1-f}{n}\right)C_z^2,$$

$$B_{5(srs)}=\left(\frac{1-f}{n}\right)\rho_{xz}C_zC_x,$$

$$B_{6(srs)}=\left(\frac{1-f}{n}\right)\frac{C_x^2}{2},$$

$$B_{7(srs)}=\left(\frac{1-f}{n}\right)\frac{C_z^2}{2}$$

$\left(B_{3(srs)} \text{ and } B_{4(srs)}\right)$ are same as defined in theorem 3.1 earlier.

To obtain the optimal values of the weights w_4, w_5 and w_6 the mean square error expression given in equation (45) is minimized with respect to these parameters. Since the MSE is a quadratic function of the weights, the optimal values can be obtained by applying the method of partial differentiation.

Specifically, the first-order partial derivatives of $MSE\left(\hat{\hat{Y}}_{10}\right)$ with respect to w_4, w_5 and w_6 are taken and equated to zero. This yields the following system of normal equations:

$$\frac{\partial MSE\left(\hat{\hat{Y}}_{10}\right)}{\partial w_4}=0,\ \frac{\partial MSE\left(\hat{\hat{Y}}_{10}\right)}{\partial w_5}=0,\ \frac{\partial MSE\left(\hat{\hat{Y}}_{10}\right)}{\partial w_6}=0$$

Solving these equations simultaneously leads to the linear system

$$\begin{bmatrix} B_{0(srs)} & B_{3(srs)} & B_{4(srs)} \\ B_{3(srs)} & B_{1(srs)} & B_{5(srs)} \\ B_{4(srs)} & B_{5(srs)} & B_{2(srs)} \end{bmatrix}\begin{bmatrix} w_4 \\ w_5 \\ w_6 \end{bmatrix}=\begin{bmatrix} 1 \\ B_{6(srs)} \\ B_{7(srs)} \end{bmatrix} \tag{46}$$

After simplification of (46), we get the optimum values of $\left(w_4, w_5, w_6\right)$ respectively as

$$w_4=\frac{\Delta_0}{\Delta},\quad w_5=\frac{\Delta_1}{\Delta},\quad w_6=\frac{\Delta_2}{\Delta} \tag{47}$$

where

$$\Delta=\begin{bmatrix} B_{0(srs)} & B_{3(srs)} & B_{4(srs)} \\ B_{3(srs)} & B_{1(srs)} & B_{5(srs)} \\ B_{4(srs)} & B_{5(srs)} & B_{2(srs)} \end{bmatrix}$$

$$=B_{0(srs)}\left(B_{1(srs)}B_{2(srs)}-B_{5(srs)}^2\right)-B_{3(srs)}\left(B_{2(srs)}B_{3(srs)}- B_{4(srs)}B_{5(srs)}\right)+B_{4(srs)}\left(B_{3(srs)}B_{5(srs)}-B_{1(srs)}B_{4(srs)}\right)$$

$$\Delta_0=\begin{bmatrix} 1 & B_{3(srs)} & B_{4(srs)} \\ B_{6(srs)} & B_{1(srs)} & B_{5(srs)} \\ B_{7(srs)} & B_{5(srs)} & B_{2(srs)} \end{bmatrix}$$

$$=\left(B_{1(srs)}B_{2(srs)}-B_{5(srs)}^2\right)-B_{3(srs)}\left(B_{2(srs)}B_{6(srs)}- B_{5(srs)}B_{7(srs)}\right)+B_{4(srs)}\left(B_{5(srs)}B_{6(srs)}-B_{1(srs)}B_{7(srs)}\right)$$

$$\Delta_1 = \begin{bmatrix} B_{0(srs)} & 1 & B_{4(srs)} \\ B_{3(srs)} & B_{6(srs)} & B_{5(srs)} \\ B_{4(srs)} & B_{7(srs)} & B_{2(srs)} \end{bmatrix}$$

$$= B_{0(srs)}\left(B_{2(srs)}B_{6(srs)} - B_{5(srs)}B_{7(srs)}\right) - \left(B_{2(srs)}B_{3(srs)} - B_{4(srs)}B_{5(srs)}\right) + B_{4(srs)}\left(B_{3(srs)}B_{7(srs)} - B_{4(srs)}B_{6(srs)}\right)$$

$$\Delta_2 = \begin{bmatrix} B_{0(srs)} & B_{3(srs)} & 1 \\ B_{3(srs)} & B_{1(srs)} & B_{6(srs)} \\ B_{4(srs)} & B_{5(srs)} & B_{7(srs)} \end{bmatrix}$$

$$= B_{0(srs)}\left(B_{1(srs)}B_{7(srs)} - B_{5(srs)}B_{6(srs)}\right) - B_{3(srs)}\left(B_{3(srs)}B_{7(srs)} - B_{4(srs)}B_{6(srs)}\right) + \left(B_{3(srs)}B_{5(srs)} - B_{4(srs)}B_{6(srs)}\right)$$

Substitution of (47) in (45) yields the minimum MSE of $\hat{\hat{Y}}_{10}$ as

$$MSE\left(\hat{\hat{Y}}_{10}\right)_{min.} = \bar{Y}^2\left[1 - \frac{\Delta_0}{\Delta} - \frac{B_{6(srs)}\Delta_1}{\Delta} - \frac{B_{7(srs)}\Delta_2}{\Delta}\right]. \tag{48}$$

4. EFFICIENCY COMPARISON

The execution of the proposed estimators is done by algebraic comparison of the MSEs of the proposed estimator and the commonly used estimators. The proposed estimator outperforms the commonly used estimators under the below algebraic conditions.

Theorem 4.1

The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than the usual unbiased estimator $\hat{\hat{Y}}_{0}$

If $\left(\frac{1-f}{n}\right)C_y^2 > \left[1 - \frac{\Delta_0}{\Delta} - \frac{B_{6(srs)}\Delta_1}{\Delta} - \frac{B_{7(srs)}\Delta_2}{\Delta}\right].$

Proof: The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than the usual unbiased estimator $\hat{\hat{Y}}_{0}$

If $MSE\left(\hat{\hat{Y}}_{10}\right)_{min.} < MSE\left(\hat{\hat{Y}}_{0}\right)$

$$\Rightarrow \bar{Y}^2\left[1 - \frac{\Delta_0}{\Delta} - \frac{B_{6(srs)}\Delta_1}{\Delta} - \frac{B_{7(srs)}\Delta_2}{\Delta}\right] < \left(\frac{1-f}{n}\right)\bar{Y}^2 C_y^2$$

$$\Rightarrow \left(\frac{1-f}{n}\right)C_y^2 > \left[1 - \frac{\Delta_0}{\Delta} - \frac{B_{6(srs)}\Delta_1}{\Delta} - \frac{B_{7(srs)}\Delta_2}{\Delta}\right]$$

(Proved)

Theorem 4.2

The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than the usual ratio estimator $\hat{\hat{Y}}_{1}$

If

$$\left(\frac{1-f}{n}\right)\left[C_y^2 + C_x^2\left(1 - 2K_{yx}\right)\right] > \left[1 - \frac{\Delta_0}{\Delta} - \frac{B_{6(srs)}\Delta_1}{\Delta} - \frac{B_{7(srs)}\Delta_2}{\Delta}\right].$$

Proof: The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than the usual ratio estimator $\hat{\hat{Y}}_{1}$

If $MSE\left(\hat{\hat{Y}}_{10}\right)_{min.} < MSE\left(\hat{\hat{Y}}_{1}\right)$

$$\Rightarrow \bar{Y}^2\left[1 - \frac{\Delta_0}{\Delta} - \frac{B_{6(srs)}\Delta_1}{\Delta} - \frac{B_{7(srs)}\Delta_2}{\Delta}\right] < \left(\frac{1-f}{n}\right)\bar{Y}^2\left[C_y^2 + C_x^2\left(1 - 2K_{yx}\right)\right]$$

$$\Rightarrow \left(\frac{1-f}{n}\right)\left[C_y^2 + C_x^2\left(1 - 2K_{yx}\right)\right] > \left[1 - \frac{\Delta_0}{\Delta} - \frac{B_{6(srs)}\Delta_1}{\Delta} - \frac{B_{7(srs)}\Delta_2}{\Delta}\right]$$

(Proved)

Theorem 4.3

The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than the usual product estimator $\hat{\hat{Y}}_{2}$

If

$$\left(\frac{1-f}{n}\right)\left[C_y^2 + C_x^2\left(1 + 2K_{yx}\right)\right] > \left[1 - \frac{\Delta_0}{\Delta} - \frac{B_{6(srs)}\Delta_1}{\Delta} - \frac{B_{7(srs)}\Delta_2}{\Delta}\right].$$

Proof: The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than the usual product estimator $\hat{\hat{Y}}_{2}$

If $MSE\left(\hat{\hat{Y}}_{10}\right)_{min.} < MSE\left(\hat{\hat{Y}}_{2}\right)$

$$\Rightarrow \bar{Y}^2\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]<\left(\frac{1-f}{n}\right)\bar{Y}^2\left[C_y^2+C_x^2\left(1+2K_{yx}\right)\right]$$

$$\Rightarrow\left(\frac{1-f}{n}\right)\left[C_y^2+C_x^2\left(1+2K_{yx}\right)\right]>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]$$

(Proved)

Theorem 4.4

The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than the estimator $\hat{\hat{Y}}_3$

If

$$\left(\frac{1-f}{n}\right)C_y^2\left[1-\rho_{yx}^2\right]>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right].$$

Proof: The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than the estimator $\hat{\hat{Y}}_3$

If $MSE\left(\hat{\hat{Y}}_{10}\right)_{min.} < MSE\left(\hat{\hat{Y}}_3\right)_{min.}$

$$\Rightarrow \bar{Y}^2\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]<\left(\frac{1-f}{n}\right)\bar{Y}^2C_y^2\left[1-\rho_{yx}^2\right]$$

$$\Rightarrow\left(\frac{1-f}{n}\right)C_y^2\left[1-\rho_{yx}^2\right]>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]$$

(Proved)

Theorem 4.5

The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than the traditional difference estimator $\hat{\hat{Y}}_4$

If

$$\left(\frac{1-f}{n}\right)C_y^2\left[1-\rho_{yx}^2\right]>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right].$$

Proof: The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than the traditional difference estimator $\hat{\hat{Y}}_5$

If $MSE\left(\hat{\hat{Y}}_{10}\right)_{min.} < MSE\left(\hat{\hat{Y}}_4\right)_{min.}$

$$\Rightarrow \bar{Y}^2\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]<\left(\frac{1-f}{n}\right)\bar{Y}^2C_y^2\left[1-\rho_{yx}^2\right]$$

$$\Rightarrow\left(\frac{1-f}{n}\right)C_y^2\left[1-\rho_{yx}^2\right]>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]$$

(Proved)

Theorem 4.6

The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than Upadhaya *et al.* (1985) class of estimators $\hat{\hat{Y}}_5$

If

$$\left[1-\frac{\left\{A_{1(srs)}-2A_{3(srs)}A_{6(srs)}+A_{0(srs)}A_{6(srs)}^2\right\}}{\left(A_{0(srs)}A_{1(srs)}-A_{3(srs)}^2\right)}\right]>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right].$$

Proof: The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than the traditional difference estimator $\hat{\hat{Y}}_5$

If $MSE\left(\hat{\hat{Y}}_{10}\right)_{min.} < MSE\left(\hat{\hat{Y}}_5\right)_{min.}$

$$\Rightarrow \bar{Y}^2\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]<\bar{Y}^2\left[1-\frac{\left\{A_{1(srs)}-2A_{3(srs)}A_{6(srs)}+A_{0(srs)}A_{6(srs)}^2\right\}}{\left(A_{0(srs)}A_{1(srs)}-A_{3(srs)}^2\right)}\right]$$

$$\Rightarrow\left[1-\frac{\left\{A_{1(srs)}-2A_{3(srs)}A_{6(srs)}+A_{0(srs)}A_{6(srs)}^2\right\}}{\left(A_{0(srs)}A_{1(srs)}-A_{3(srs)}^2\right)}\right]>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]$$

(Proved)

Theorem 4.7

The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than the class of estimators $\hat{\hat{Y}}_5^*$

If

$$\left(\frac{1-f}{n}\right)C_y^2\left[1-\rho_{yx}^2\right]>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right].$$

Proof: The proposed estimator $\hat{\hat{\bar{Y}}}_{10}$ is more efficient than the class of estimators $\hat{\hat{\bar{Y}}}_5^*$

If $MSE\left(\hat{\hat{\bar{Y}}}_{10}\right)_{min.} < MSE\left(\hat{\hat{\bar{Y}}}_5^*\right)_{min.}$

$$\Rightarrow \bar{Y}^2\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]<\left(\frac{1-f}{n}\right)\bar{Y}^2C_y^2\left[1-\rho_{yx}^2\right]$$

$$\Rightarrow\left(\frac{1-f}{n}\right)C_y^2\left[1-\rho_{yx}^2\right]>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]$$

(Proved)

Theorem 4.8

The proposed estimator $\hat{\hat{\bar{Y}}}_{10}$ is more efficient than the difference- type estimator $\hat{\hat{\bar{Y}}}_6$

If

$$\frac{\left(\frac{1-f}{n}\right)\left(C_y^2-K_{yx}^2C_x^2\right)}{1+\left(\frac{1-f}{n}\right)\left(C_y^2-K_{yx}^2C_x^2\right)}>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right].$$

Proof: The proposed estimator $\hat{\hat{\bar{Y}}}_{10}$ is more efficient than the difference- type estimator $\hat{\hat{\bar{Y}}}_6$

If $MSE\left(\hat{\hat{\bar{Y}}}_{10}\right)_{min.} < MSE\left(\hat{\hat{\bar{Y}}}_6\right)_{min.}$

$$\Rightarrow \bar{Y}^2\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]<\frac{\left(\frac{1-f}{n}\right)\bar{Y}^2\left(C_y^2-K_{yx}^2C_x^2\right)}{1+\left(\frac{1-f}{n}\right)\left(C_y^2-K_{yx}^2C_x^2\right)}$$

$$\Rightarrow\frac{\left(\frac{1-f}{n}\right)\left(C_y^2-K_{yx}^2C_x^2\right)}{1+\left(\frac{1-f}{n}\right)\left(C_y^2-K_{yx}^2C_x^2\right)}>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]$$

(Proved)

Theorem 4.9

The proposed estimator $\hat{\hat{\bar{Y}}}_{10}$ is more efficient than Gupta and Shabbir (2008) class of estimators $\hat{\hat{\bar{Y}}}_7$

If

$$\left\{1-\frac{\left(a_1a_4^2-2a_3a_4a_5+a_1a_5^2\right)}{\left(a_1a_2-a_3^2\right)}\right\}>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right].$$

Proof: The proposed estimator $\hat{\hat{\bar{Y}}}_{10}$ is more efficient than Gupta and Shabbir (2008) class of estimators $\hat{\hat{\bar{Y}}}_7$

If $MSE\left(\hat{\hat{\bar{Y}}}_{10}\right)_{min.} < MSE\left(\hat{\hat{\bar{Y}}}_7\right)_{min.}$

$$\Rightarrow \bar{Y}^2\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]< \bar{Y}^2\left\{1-\frac{\left(a_1a_4^2-2a_3a_4a_5+a_1a_5^2\right)}{\left(a_1a_2-a_3^2\right)}\right\}$$

$$\Rightarrow\left\{1-\frac{\left(a_1a_4^2-2a_3a_4a_5+a_1a_5^2\right)}{\left(a_1a_2-a_3^2\right)}\right\}> \left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]$$

(Proved)

Theorem 4.10

The proposed estimator $\hat{\hat{\bar{Y}}}_{10}$ is more efficient than traditional difference estimator $\hat{\hat{\bar{Y}}}_8$ using two auxiliary variables x and z

If

$$\left(\frac{1-f}{n}\right)C_y^2\left[1-R_{y.xz}^2\right]>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right].$$

Proof: The proposed estimator $\hat{\hat{\bar{Y}}}_{10}$ is more efficient than traditional difference estimator $\hat{\hat{\bar{Y}}}_8$ using two auxiliary variables x and z

If $MSE\left(\hat{\hat{\bar{Y}}}_{10}\right)_{min.} < MSE\left(\hat{\hat{\bar{Y}}}_8\right)_{min.}$

$$\Rightarrow \bar{Y}^2\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]<\left(\frac{1-f}{n}\right)\bar{Y}^2C_y^2\left[1-R_{y.xz}^2\right]$$

$$\Rightarrow\left(\frac{1-f}{n}\right)C_y^2\left[1-R_{y.xz}^2\right]>\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]$$

(Proved)

Theorem 4.11

The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than Singh and Nigam (2022) class of estimators $\hat{\hat{Y}}_9$

If

$$\left[\frac{\Delta_0^{**}}{\Delta^{**}}+\frac{A_{6(srs)}\Delta_1^{**}}{\Delta^{**}}+\frac{A_{7(srs)}\Delta_2^{**}}{\Delta^{**}}\right]>\left[\frac{\Delta_0}{\Delta}+\frac{B_{6(srs)}\Delta_1}{\Delta}+\frac{B_{7(srs)}\Delta_2}{\Delta}\right].$$

Proof: The proposed estimator $\hat{\hat{Y}}_{10}$ is more efficient than Singh and Nigam (2022) class of estimators $\hat{\hat{Y}}_9$

If $MSE\left(\hat{\hat{Y}}_{10}\right)_{min.} < MSE\left(\hat{\hat{Y}}_9\right)_{min.}$

$$\Rightarrow \bar{Y}^2\left[1-\frac{\Delta_0}{\Delta}-\frac{B_{6(srs)}\Delta_1}{\Delta}-\frac{B_{7(srs)}\Delta_2}{\Delta}\right]<$$

$$\bar{Y}^2\left[1-\frac{\Delta_0^{**}}{\Delta^{**}}-\frac{A_{6(srs)}\Delta_1^{**}}{\Delta^{**}}-\frac{A_{7(srs)}\Delta_2^{**}}{\Delta^{**}}\right]$$

$$\Rightarrow\left[\frac{\Delta_0^{**}}{\Delta^{**}}+\frac{A_{6(srs)}\Delta_1^{**}}{\Delta^{**}}+\frac{A_{7(srs)}\Delta_2^{**}}{\Delta^{**}}\right]>\left[\frac{\Delta_0}{\Delta}+\frac{B_{6(srs)}\Delta_1}{\Delta}+\frac{B_{7(srs)}\Delta_2}{\Delta}\right]$$

(Proved)

5. EMPIRICAL STUDIES

To examine the performance of the proposed estimator, three real data sets that are frequently used in the survey sampling literature were considered. These datasets contain a study variable and two auxiliary variables with known population parameters and correlation structures, making them suitable for evaluating the efficiency of different estimators.

Data Set I is taken from Singh and Chaudhary (1986) and consists of agricultural survey data. In this dataset, the study variable represents the primary variable of interest, while the auxiliary variables correspond to related agricultural characteristics that are correlated with the study variable.

Data Set II is obtained from Cochran (1977), which is a widely used reference in sampling theory. The dataset contains variables that exhibit moderate to strong correlation, making it useful for examining the performance of ratio-type and regression-type estimators using auxiliary information.

Data Set III is taken from Ahmed (1977) and has been widely used in studies related to ratio and chain estimators in survey sampling. The variables in this dataset also demonstrate significant correlation between the study variable and auxiliary variables, which makes it appropriate for evaluating the efficiency gains obtained by incorporating multiple auxiliary variables.

These datasets are selected because they are well-established benchmark datasets in the survey sampling literature and provide different correlation structures between the study and auxiliary variables. This allows a meaningful comparison of the proposed estimator with existing estimators under realistic conditions.

Data	I	II	III
N	34	34	376
n	20	15	159
$\bar{Y}$	856.41	4.92	316.65
$\bar{X}$	208.88	2.59	141.13
$\bar{Z}$	199.44	2.91	1075.31
C_y	0.85	1.01232	0.7721
C_x	0.72	1.23187	0.845
C_z	0.75	1.05351	0.7746
ρ_{yx}	0.45	0.7326	0.9106
ρ_{yz}	0.45	0.643	0.9094
ρ_{xz}	0.98	0.6837	0.8614

Table 1 gives the PRE's of $\hat{\hat{Y}}_0$, $\hat{\hat{Y}}_1$, $\hat{\hat{Y}}_2$, $\hat{\hat{Y}}_3$, $\hat{\hat{Y}}_4$, $\hat{\hat{Y}}_5^{\,*}$, $\hat{\hat{Y}}_6$, $\hat{\hat{Y}}_7$ and $\hat{\hat{Y}}_8$ estimators with respect to $\bar{y}$ for three data sets respectively.

Table 2 gives the PRE's of $\hat{\hat{Y}}_5$ with respect to $\bar{y}$ for $\alpha_1=(-1,1)$, for three data sets.

Table 3 gives the PRE's of $\hat{\hat{Y}}_9$ with respect to $\bar{y}$ for $(\alpha_1,\alpha_2)=(-1,-1)$ and $(\alpha_1,\alpha_2)=(1,1)$, for three data sets.

Table 1. PRE's of different estimators of population mean $\bar{Y}$ with respect to $\bar{y}$

Estimator	Data I	Data II	Data III
$\hat{\bar{Y}}_0$	100	100	100
$\hat{\bar{Y}}_1$	105.55	143.3	488.77
$\hat{\bar{Y}}_2$	40.74	23.45	23.86
$\hat{\bar{Y}}_3$	125.39	215.84	585.45
$\hat{\bar{Y}}_4$	125.39	215.84	585.45
$\hat{\bar{Y}}_5^*$	125.39	215.84	585.45
$\hat{\bar{Y}}_6$	126.92	219.66	585.67
$\hat{\bar{Y}}_7$	126.93	219.89	585.67
$\hat{\bar{Y}}_8$	125.71	235.09	907.16

Table 2. PRE's of $\hat{\bar{Y}}_5$ with respect to $\bar{y}$

α_1	Data I	Data II	Data III
-1	127.66	227.99	586.30
1	126.24	215.95	588.70

Table 3. PRE's of $\hat{\bar{Y}}_9$ with respect to $\bar{y}$

α_1	α_2	Data I	Data II	Data III
-1	-1	128.03	248.82	908.24
1	1	126.56	235.59	918.17

Table 4 indicates the PRE of proposed estimator $\hat{\bar{Y}}_{10}$ with respect to $\bar{y}$ for three data sets.

Table 4. indicates the PRE of proposed estimator $\hat{\bar{Y}}_{10}$ with respect to $\bar{y}$

Estimator	Data I	Data II	ta III
$\hat{\bar{Y}}_{10}$	**200.713**	**303.376**	**4347.797**

We measure the Percent Relative Efficiencies (PREs) of several estimators, including the proposed estimator $\hat{\bar{Y}}_{10}$, with respect to $\bar{y}$. Tables 1, 2, 3 and 4 show that the suggested estimator $\hat{\bar{Y}}_{10}$ provides the highest PRE (200.713%, 303.376% and 4347.797%) for the data sets I-III, respectively. Using the proposed estimator $\hat{\bar{Y}}_{10}$ over the existing estimator results in a significant increase in efficiency. When these auxiliary variables are strongly correlated with the study variable, the variance of the estimator is significantly reduced, leading to higher PRE values. The large PRE for the Data Set III (4347.797%) is mainly due to the strong correlation between the study and auxiliary variables. Thus, the proposed estimator $\hat{\bar{Y}}_{10}$ can be effectively utilized in practical applications.

6. CONCLUSION

This article explores calculating the population mean $\bar{Y}$. of a study variable using information from two auxiliary variables (x, z). Bias and mean square error expressions up to the first order of approximation were obtained using SRSWOR sampling. It's worth noting that the envisioned class of estimators contains some existing estimators. Thus, the features of the proposed estimator bring results together in one place. The suggested estimators outperform existing estimators in SRSWOR sampling designs, according to theoretical analysis. Empirical studies are conducted to assess the advantages of the proposed estimator against existing competitors. The suggested estimator outperforms other estimators in SRSWOR sampling, resulting in higher efficiency. The findings of these studies are both reliable and insightful.

The application of developing an extensive variety of estimator for the finite population mean in sample surveys using two auxiliary variables lies primarily in enhancing the accuracy and efficiency of statistical estimates across various fields. By incorporating two easily measurable or known auxiliary variables such as age and income, rainfall and fertilizer use, or electricity consumption and household size researchers and policymakers can obtain more reliable estimates of the population mean without increasing sample size or cost. This approach is especially valuable in large-scale national surveys, agricultural studies, public health assessments, and market research, where precise estimation is crucial for planning, policy formulation, and resource allocation. Utilizing two auxiliary variables allows for the construction of estimators that minimize bias and variance, thereby improving the overall quality of survey-based inferences in both governmental and private sector applications.

The proposed estimator can be particularly useful in practical survey situations where reliable auxiliary information is available. For example, in agricultural surveys, crop yield (study variable) can be estimated more precisely using auxiliary variables such as cultivated land area and fertilizer usage. In public health studies, the average healthcare expenditure of households may be estimated using auxiliary variables like household income and family size. Similarly, in economic and socio-economic surveys, variables such as consumption expenditure or employment levels can be estimated using auxiliary information such as income level and education status. By incorporating multiple auxiliary variables simultaneously, the proposed estimator can significantly improve the precision of population mean estimation.

Overall, the findings suggest that the proposed estimator provides an alternative for estimating the population mean when auxiliary information is available. Future research can extend this approach to other sampling schemes and examine its performance under various sampling designs.

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