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Study of Some Generalized Classes of Estimators in Probability Proportional to Size Sampling : An Empirical Study

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SUMMARY

The aim of this study is to propose three novel generalized classes of estimators under probability proportional to size PPS sampling scheme. Employing a first-order approximation (foa), the expressions for the bias and mean squared error ($MSeR$) are derived. Actual dataset is used in the quantitative assessment to evaluate the competency of the new proposed class of estimators. From the outcome of numerical study, it is advocated that the proposed estimators gives the least $MSeR$ and utmost precent relative efficiency (PRE) relative to other estimators .

Keywords : Probability proportional to size (PPS); Bias; $MSeR$; Empirical Study; PRE .

1. INTRODUCTION

Estimation of finite population mean (PM) is a fundamental problem in survey sampling and significant amount of research has been devoted in improving the efficiency of estimators. Abundant approaches have been put forward to increase the precision of the estimates by utilizing auxiliary information such as ratio (Rt), product (Pt) and regression ($ReGR$) type methods. Several attempts have been made by the researchers and investigators to estimate the population parameters by incorporating the finest statistical parameters such as variance, median, coefficient of kurtosis etc. associated with auxiliary/study variables. Simple random sampling (SRS) with or without replacement is used when the population under study holds similarity and a representative sample of adequate size is drawn. Many investigators by utilizing the auxiliary information studied numerous estimators under different sampling methods, such as when there is negative correlation Robson (1957) proposed Pt estimator for PM , Bahl and Tuteja (1991) suggested exponential Rt and Pt type estimators, Yadav and Kadilar (2013) also developed an efficient family of exponential estimators for PM . For estimating the mean of a study variable Prasad (1989) also studied some improved Rt type estimators, mathematical characteristics of different Rt and $ReGR$ estimators were evaluated by Singh and Espejo (2003), Dalabehra and Sahoo (1996) proposed class of estimators in stratified sampling. Kadilar and Cingi (2003) studied Rt estimators, Kiregyera (1984) proposed regression type estimators in double sampling. Agrawal and Panda (1993) suggest an efficient estimator in post stratification scheme. Optimality of Rt type imputation methods for PM were observed by Bhushan and Pandey (2021), a generalized class of estimators of estimators was studied by Ahmad *et al.* (2022) in two - stage sampling. {Patel *et al.* (2019) proposed a class of Rt – type estimators under two – phase sampling scheme using auxiliary variables. Misra *et al.* (2016) suggested improved estimators for estimating variance of the population in presence of measurement errors. $ReGR$ type estimator is proposed by Grover *et al.* (2020) in stratified double sampling. When the population units differ significantly in size then the most appropriate

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method of estimation is PPS scheme, where the probability of inclusion of each unit is proportional to auxiliary variable (Av) which is correlated with the study variable (Sv). This design permits the greater representation of influential units in the sample which results in the least error and enhanced precision. The PPS method is most appropriate in many real-life situations for instance, in household income survey, the probability of selecting a household may depend on the number of family members; in agricultural study, farm and villages are often chosen in proportion to their cultivated land area or livestock count. Also, in health sector, hospitals are frequently selected based on the number of available beds or registered patients. Different estimators are studied by researchers under PPS scheme like Wang *et al.* (2023) studied estimators of finite population in presence of extreme values in PPS sampling, Patel *et al.* (2016) proposed estimator for population total in presence of extra auxiliary information. Al - Marzouki *et al.* (2021) also discussed the estimator for PM in presence of maximum and minimum values, Ahmad *et al.* (2018) studied the estimator PM using extreme values under PPS scheme, Sohil *et al.* (2022) studies imputation procedure in PPS sampling.

2. PROCEDURE AND NOTATIONS

Consider a population of finite size N, comprises of units $\omega=\{\omega_1,\omega_2,\ldots\ldots,\omega_N\}$. Let y_i and $\{x_i,z_i\}$ be the values of study variable (Y) and the supplementary variable (X, Z) respectively. We pull out a sample of size n by applying PPS sampling acquiring Z as a size of the unit, i.e., $K_i=\frac{z_i}{\sum_{i=1}^{N} z_i}$, be the PPS scheme for obtaining the units. We draw a sample of size n under PPS scheme with replacement.

We define

$$u_i=\frac{y_i}{NK_i},\quad v_i=\frac{x_i}{NK_i}.$$

Sample means corresponding to the population mean $\bar{Y}$ and $\bar{X}$ can be defined as

$$\bar{u}=\frac{\sum_{i=1}^{n} u_i}{n}=\bar{y}_{pps},\quad \bar{v}=\frac{\sum_{i=1}^{n} v_i}{n}=\bar{x}_{pps}.$$

For obtaining the bias and $MSeR$ of the estimators, we consider the following error terms

Let, $\bar{u}=\bar{Y}(1+\psi_0)$ and $\bar{v}=\bar{X}(1+\psi_1)$, such that

$$\psi_0=\frac{\bar{u}-\bar{Y}}{\bar{Y}},\ \psi_1=\frac{\bar{v}-\bar{X}}{\bar{X}},\ E(\psi_i)=0, i=0,1;$$

$$E(\psi_0^2)=\lambda C_u^2,\ E(\psi_0^2)=\lambda C_v^2,\ E(\psi_0\psi_1)=\lambda\rho_{uv}C_uC_v,$$

here $\lambda=\frac{1}{n}$

$C_u=\frac{S_u}{\bar{Y}}$, $C_v=\frac{S_v}{\bar{X}}$, be the population coefficients of variation (CoV) of the variables (u,v).

$$S_u=\sqrt{\sum_{i=1}^{N} K_i(u_i-\bar{Y})^2},\ S_v=\sqrt{\sum_{i=1}^{N} K_i(v_i-\bar{X})^2}.$$

Let the correlation coefficient (Cc) of uand v can be defined as

$$\rho_{uv}=\frac{\sum_{i=1}^{N} K_i(u_i-\bar{Y})(v_i-\bar{X})}{S_uS_v}.$$

3. OVERVIEW OF EXISTING METHODS

This section discusses some traditional and existing methods of estimation available in the context of PPS sampling,

(i) Under PPS, the classical estimator is given by

$$\hat{\hat{A}}_{(U,PPS)}=\bar{u}, \tag{3.1}$$

The variance associated with $\hat{\hat{A}}_{(U,PPS)}$ is given by

$$Var\left(\hat{\hat{A}}_{(U,PPS)}\right)=\lambda\bar{Y}^2C_u^2. \tag{3.2}$$

(ii) The Rt estimator under PPS scheme, is given by

$$\hat{\hat{A}}_{(r,PPS)}=\left(\frac{\bar{X}}{\bar{v}}\right)\bar{u}. \tag{3.3}$$

The $MSeR$ of $\hat{\hat{A}}_{(r,PPS)}$ up to foa is given by

$$MSE\left(\hat{\hat{A}}_{(r,PPS)}\right)=\lambda\bar{Y}^2\left[C_u^2+C_v^2-2\rho_{uv}C_uC_v\right]. \tag{3.4}$$

(iii) Murthy (1967) introduced the Pt estimator, which is expressed as

$$\hat{\hat{A}}_{(p,PPS)}=\bar{u}\left(\frac{\bar{v}}{\bar{X}}\right). \tag{3.5}$$

The $MSeR$ of $\hat{\hat{A}}_{(p,PPS)}$ up to *foa* is obtained as

$$MSE\left(\hat{\hat{A}}_{(p,PPS)}\right)=\lambda\bar{Y}^2\left[C_u^2+C_v^2+2\rho_{uv}C_uC_v\right]. \quad (3.6)$$

(iv) The $ReGR$ type estimator is defined as

$$\hat{\hat{A}}_{(reg,PPS)}=\bar{u}+\varphi_1\left(\bar{X}-\bar{v}\right) \quad (3.7)$$

where $,\varphi_1$ is constant.

The optimal form of $MSeR$ of $\hat{\hat{A}}_{(reg,PPS)}$ is given by

$$MSE_{min}\left(\hat{\hat{A}}_{(reg,PPS)}\right)=C_u^2\lambda\bar{Y}^2\left(1-\rho_{uv}^2\right). \quad (3.8)$$

(v) The estimator due toBai *et al.* (1991) is represented by

$$\hat{\hat{A}}_{(Bai,PPS)}=\varphi_2\bar{u}+\varphi_3\left(\bar{X}-\bar{v}\right), \quad (3.9)$$

where, (φ_2,φ_3) corresponds to unknown constants.

The minimum $MSeR$ of $\hat{\hat{A}}_{(Bai,PPS)}$ is given by

$$MSE_{min}\left(\hat{\hat{A}}_{(Bai,PPS)}\right)=\frac{\lambda\bar{Y}^2C_u^2\left(1-\rho_{uv}^2\right)}{\left\{1+\lambda C_u^2\left(1-\rho_{uv}^2\right)\right\}}. \quad (3.10)$$

(vi) The Rt and Pt exponential type estimators on the line of Bahl and Tuteja (1991), are respectively defined by

$$\hat{\hat{A}}_{(BTr,PPS)}=\bar{u}\,exp\left(\frac{\bar{X}-\bar{v}}{\bar{X}+\bar{v}}\right), \quad (3.11)$$

$$\hat{\hat{A}}_{(BTp,PPS)}=\bar{u}\,exp\left(\frac{\bar{v}-\bar{X}}{\bar{v}+\bar{X}}\right). \quad (3.12)$$

The MSE's of $\hat{\hat{Y}}_{(BTr,PPS)}$ and $\hat{\hat{Y}}_{(BTr,PPS)}$ to the *foa* are respectively given by

$$MSE(\hat{\hat{A}}_{(BTr,PPS)})=\lambda\bar{Y}^2\left(\frac{1}{4}C_v^2-\rho_{uv}C_uC_v+C_u^2\right), \quad (3.13)$$

$$MSE(\hat{\hat{A}}_{(BTp,PPS)})=\lambda\bar{Y}^2\left(\frac{1}{4}C_v^2+\rho_{uv}C_uC_v+C_u^2\right). \quad (3.14)$$

(vii) Following estimators were formulated by Haq and Shabbir (2014)

$$\hat{\hat{A}}_{(HS1,PPS)}=\left\{\frac{\varphi_4}{2}\bar{u}\left(\frac{\bar{X}}{\bar{v}}+\frac{\bar{v}}{\bar{X}}\right)+\varphi_5\left(\bar{X}-\bar{v}\right)\right\}exp\left(\frac{\bar{X}-\bar{v}}{\bar{X}+\bar{v}}\right), \quad (3.15)$$

where, φ_4 and φ_5 are constants.

The optimum $MSeR$ of $\hat{\hat{A}}_{(HS1,PPS)}$ is given by

$$MSE_{min}\left(\hat{\hat{A}}_{(HS1,PPS)}\right)=\bar{Y}^2\left[1-\left(\frac{A_{H1}D_{H1}^2+B_{H1}C_{H1}^2-C_{H1}D_{H1}E_{H1}}{A_{H1}B_{H1}-E_{H1}^2}\right)\right], \quad (3.16)$$

where $A_{H1}=1+\lambda\left[C_u^2+2C_v^2-2\rho_{uv}C_uC_v\right]$,

$B_{H1}=R^2\lambda C_v^2$,

$C_{H1}=1+\lambda\left[\frac{7}{8}C_v^2-\frac{1}{2}\rho_{uv}C_uC_v\right]$,

$D_{H1}=R\lambda\frac{C_v^2}{2}$,

$E_{H1}=R\lambda\left[C_v^2-\rho_{uv}C_uC_v\right],\ R=\frac{\bar{Y}}{\bar{X}}$.

Another estimator $\hat{\hat{A}}_{(HS2,PPS)}$ discussed by Haq and Shabbir (2014), is given by

$$\hat{\hat{A}}_{(HS2,PPS)}=\left\{\varphi_6\hat{\hat{Y}}_{HSA,PPS}+\varphi_7\left(\bar{X}-\bar{v}\right)\right\}exp\left(\frac{\bar{X}-\bar{v}}{\bar{X}+\bar{v}}\right), \quad (3.17)$$

where $(\varphi_6,\ \varphi_7)$ are constants, and

$$\hat{\hat{A}}_{(HSA,PPS)}=\frac{\bar{u}}{2}\left[exp\left(\frac{\bar{X}-\bar{v}}{\bar{X}+\bar{v}}\right)+exp\left(\frac{\bar{v}-\bar{X}}{\bar{v}+\bar{X}}\right)\right].$$

The optimal $MSeR$ of $\hat{\hat{A}}_{(HS2,PPS)}$ is given by

$$MSE_{min}\left(\hat{\hat{A}}_{(HS2,PPS)}\right)=\bar{Y}^2\left[1-\left(\frac{A_{H2}D_{H2}^2+B_{H2}C_{H2}^2-2C_{H2}D_{H2}E_{H2}}{A_{H2}B_{H2}-E_{H2}^2}\right)\right] \quad (3.18)$$

where $A_{H2}=1+\lambda\left[C_u^2+\frac{5}{4}C_v^2-2\rho_{uv}C_uC_v\right]$,

$B_{H2}=\lambda C_v^2R^2$,

$C_{H2}=1\left[\frac{1}{2}C_v^2-\frac{1}{2}\rho_{uv}C_uC_v\right]+\lambda$,

$D_{H2}=R\lambda\frac{C_v^2}{2}$,

$E_{H2}=R\lambda\left[C_v^2-\rho_{uv}C_u\right]$.

(viii) Ekpenyong *et al.* (2015) advocated the following estimator

$$\hat{\hat{A}}_{(E,PPS)}=\varphi_8\bar{u}+\varphi_9\left(\bar{X}-\bar{v}\right)exp\left(\frac{\bar{X}-\bar{v}}{\bar{X}+\bar{v}}\right), \quad (3.19)$$

where φ_8 and φ_9 are constants.

The lowest $MSeR$ of $\hat{\hat{A}}_{(E,PPS)}$ can be expressed as

$$MSE_{min}\left(\hat{\hat{A}}_{(E,PPS)}\right)=\bar{Y}^2\left[1-\left(\frac{A_E D_E^2+B_E C_E^2-2C_E D_E E_E}{A_E B_E-E_E^2}\right)\right], \tag{3.20}$$

where $A_E=1+\lambda C_u^2$,

$B_E=R^2\lambda C_v^2$,

$C_E=1$,

$D_E=R\lambda\dfrac{\theta^2 C_v^2}{2}$,

$E_E=R\lambda\left[\dfrac{C_v^2}{2}-\rho_{uv}C_u C_v\right]$.

(ix) Singh *et al.* (2009) presented the following class of estimators

$$\hat{\hat{A}}^*_{(S,PPS)}=\bar{u}\,exp\left(\frac{a\left(\bar{X}-\bar{v}\right)}{a\left(\bar{X}+\bar{v}\right)+2b}\right), \tag{3.21}$$

where a and b are real numbers or known population parameters associated with Sv and Av .

The MSE of $\hat{\hat{A}}^*_{(S,PPS)}$, to the *foa* is given by

$$MSE(\hat{\hat{A}}^*_{(S,PPS)})=\frac{\bar{Y}^2\lambda}{4}\left[4C_u^2+\theta^2C_v^2-4\theta\rho_{uv}C_uC_v\right], \tag{3.22}$$

where $\theta=\dfrac{a\bar{X}}{a\bar{X}+b}$.

(x) Following generalized estimator was discussed by Grover *et al.*(2014)

$$\hat{\hat{A}}^*_{(G,PPS)}=\left[\varphi_{10}\bar{u}+\varphi_{11}\left(\bar{X}-\bar{v}\right)\right]exp\left(\frac{a\left(\bar{X}-\bar{v}\right)}{a\left(\bar{X}-\bar{v}\right)+2b}\right), \tag{3.23}$$

$(\varphi_{10},\varphi_{11})$ are constants, (a,b) are same as defined above.

The expression for optimum $MSeR$ of $\hat{\hat{A}}^*_{(G,PPS)}$ at the optimal values of φ_{10} and φ_{11}, isgiven by

Table 3.1. Few members belonging to the family of estimators $\hat{\hat{A}}^*_{(S,PPS)}$

a	b	Estimators
1	C_v	$exp\left(\dfrac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}+\bar{v}\right)+2C_v}\right)\bar{u}$
1	$\beta_{2(v)}$	$exp\left(\dfrac{\left(\bar{X}-\bar{v}\right)}{2\beta_{2(v)}+\left(\bar{X}+\bar{v}\right)}\right)\bar{u}$
$\beta_{2(v)}$	C_v	$exp\left(\dfrac{\left(\bar{X}-\bar{v}\right)\beta_{2(v)}}{\left(\bar{X}+\bar{v}\right)\beta_{2(v)}+2C_v}\right)\bar{u}$
C_v	$\beta_{2(v)}$	$exp\left(\dfrac{\left(\bar{X}-\bar{v}\right)C_v}{\left(\bar{X}+\bar{v}\right)C_v+2\beta_{2(v)}}\right)\bar{u}$
1	ρ_{uv}	$exp\left(\dfrac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}+\bar{v}\right)+2\rho_{uv}}\right)\bar{u}$
C_v	ρ_{uv}	$exp\left(\dfrac{\left(\bar{X}-\bar{v}\right)C_v}{\left(\bar{X}+\bar{v}\right)C_v+2\rho_{uv}}\right)\bar{u}$
ρ_{uv}	C_v	$exp\left(\dfrac{\rho_{uv}\left(\bar{X}-\bar{v}\right)}{\rho_{uv}\left(\bar{X}+\bar{v}\right)+2C_v}\right)\bar{u}$
$\beta_{2(v)}$	ρ_{uv}	$exp\left(\dfrac{\left(\bar{X}-\bar{v}\right)\beta_{2(v)}}{\left(\bar{X}+\bar{v}\right)\beta_{2(v)}+2\rho_{uv}}\right)\bar{u}$
ρ_{uv}	$\beta_{2(v)}$	$exp\left(\dfrac{\rho_{uv}\left(\bar{X}-\bar{v}\right)}{\rho_{uv}\left(\bar{X}+\bar{v}\right)+2\beta_{2(v)}}\right)\bar{u}$
1	$N\bar{X}$	$exp\left(\dfrac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}+\bar{v}\right)+2N\bar{X}}\right)\bar{u}$

$$MSE_{min}\left(\hat{\hat{A}}^*_{(G,PPS)}\right)=\bar{Y}^2\left[1-\left(\frac{A_k D_k^2+B_k C_k^2-2C_k D_k E_k}{A_k B_k-E_k^2}\right)\right], \tag{3.24}$$

where $A_k=1+\lambda\left[\theta^2C_v^2-2\theta\rho_{uv}C_uC_v+C_u^2\right]$,

$B_k=R^2\lambda C_v^2$,

$C_k=1+\lambda\left[\dfrac{3}{8}\theta^2C_v^2-\dfrac{1}{2}\theta\rho_{uv}C_uC_v\right]$,

$D_k=R\lambda\dfrac{C_v^2}{2}$,

$E_k=\lambda\left[\theta^2C_v^2-\theta\rho_{uv}C_uC_v\right]$.

Table 3.2. Few members belonging to the family of estimators $\hat{\hat{A}}^{*}_{(G,PPS)}$

a	b	Estimators
1	C_v	$\left[\bar{u}\varphi_{10}+\varphi_{11}\left(\bar{X}-\bar{v}\right)\right]exp\left(\frac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}-\bar{v}\right)+2C_v}\right)$
1	$\beta_{2(v)}$	$\left[\bar{u}\varphi_{10}+\varphi_{11}\left(\bar{X}-\bar{v}\right)\right]exp\left(\frac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}-\bar{v}\right)+2\beta_{2(v)}}\right)$
$\beta_{2(v)}$	C_v	$\left[\bar{u}\varphi_{10}+\varphi_{11}\left(\bar{X}-\bar{v}\right)\right]exp\left(\frac{\beta_{2(v)}\left(\bar{X}-\bar{v}\right)}{\beta_{2(v)}\left(\bar{X}-\bar{v}\right)+2C_v}\right)$
C_v	$\beta_{2(v)}$	$\left[\bar{u}\varphi_{10}+\varphi_{11}\left(\bar{X}-\bar{v}\right)\right]exp\left(\frac{C_v\left(\bar{X}-\bar{v}\right)}{C_v\left(\bar{X}-\bar{v}\right)+2\beta_{2(v)}}\right)$
1	ρ_{uv}	$\left[\bar{u}\varphi_{10}+\varphi_{11}\left(\bar{X}-\bar{v}\right)\right]exp\left(\frac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}-\bar{v}\right)+2\rho_{uv}}\right)$
C_v	ρ_{uv}	$\left[\bar{u}\varphi_{10}+\varphi_{11}\left(\bar{X}-\bar{v}\right)\right]exp\left(\frac{C_v\left(\bar{X}-\bar{v}\right)}{C_v\left(\bar{X}-\bar{v}\right)+2\rho_{uv}}\right)$
ρ_{uv}	C_v	$\left[\bar{u}\varphi_{10}+\varphi_{11}\left(\bar{X}-\bar{v}\right)\right]exp\left(\frac{\rho_{uv}\left(\bar{X}-\bar{v}\right)}{\rho_{uv}\left(\bar{X}-\bar{v}\right)+2C_v}\right)$
$\beta_{2(v)}$	ρ_{uv}	$\left[\bar{u}\varphi_{10}+\varphi_{11}\left(\bar{X}-\bar{v}\right)\right]exp\left(\frac{\beta_{2(v)}\left(\bar{X}-\bar{v}\right)}{\beta_{2(v)}\left(\bar{X}-\bar{v}\right)+2\rho_{uv}}\right)$
ρ_{uv}	$\beta_{2(v)}$	$\left[\bar{u}\varphi_{10}+\varphi_{11}\left(\bar{X}-\bar{v}\right)\right]exp\left(\frac{\rho_{uv}\left(\bar{X}-\bar{v}\right)}{\rho_{uv}\left(\bar{X}-\bar{v}\right)+2\beta_{2(v)}}\right)$
1	$N\bar{X}$	$\left[\bar{u}\varphi_{10}+\varphi_{11}\left(\bar{X}-\bar{v}\right)\right]exp\left(\frac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}-\bar{v}\right)+2N\bar{X}}\right)$

(xi) Sohaib *et al.* (2023) developed the following estimator for $\bar{Y}$:

$$\hat{\hat{A}}^{*}_{(SH,PPS)}=\left[\varphi_{18}\hat{\hat{\bar{Y}}}_A+\varphi_{19}\left(\bar{X}-\bar{v}\right)\right]\left[exp\left(\frac{a\left(\bar{X}-\bar{v}\right)}{a\left(\bar{X}+\bar{v}\right)+2b}\right)\right], \tag{3.25}$$

where

$$\hat{\hat{\bar{Y}}}_A=\bar{u}\left\{\frac{1}{4}\left(\frac{\bar{X}}{\bar{v}}+\frac{\bar{v}}{\bar{X}}\right)\left(exp\left(\frac{\bar{X}-\bar{v}}{\bar{X}+\bar{v}}\right)+exp\left(\frac{\bar{v}-\bar{X}}{\bar{v}+\bar{X}}\right)\right)\right\}$$

and

$(\varphi_{18},\varphi_{19})$ are constants, a and b aresame as defined earlier.

The least $MSeR$ of $\hat{\hat{A}}^{*}_{(SH,PPS)}$ for the optimal values of φ_{18} and φ_{19} isgiven by

$$MSE_{min}\left(\hat{\hat{A}}^{*}_{(SH,PPS)}\right)==\bar{Y}^2\left[1-\left(\frac{A_zD_z^2+B_zC_z^2-2C_zD_zE_z}{A_zB_z-E_z^2}\right)\right], \tag{3.26}$$

where $A_z=1+\lambda\left[C_u^2+\left(\frac{5}{4}+\theta^2\right)C_v^2-2\theta\rho_{uv}C_uC_v\right]$,

$B_z=R^2\lambda C_v^2$,

Table 3.3. Few members belonging to the family of estimators $\hat{\hat{\bar{Y}}}^{*}_{(SH,PPS)}$

a	b	Estimators
1	C_v	$\left[\varphi_{18}\hat{\hat{\bar{Y}}}_A+\varphi_{19}\left(\bar{X}-\bar{v}\right)\right]\left[exp\left(\frac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}+\bar{v}\right)+2C_v}\right)\right]$
1	$\beta_{2(v)}$	$\left[\varphi_{18}\hat{\hat{\bar{Y}}}_A+\varphi_{19}\left(\bar{X}-\bar{v}\right)\right]\left[exp\left(\frac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}+\bar{v}\right)+2\beta_{2(v)}}\right)\right]$
$\beta_{2(v)}$	C_v	$\left[\varphi_{18}\hat{\hat{\bar{Y}}}_A+\varphi_{19}\left(\bar{X}-\bar{v}\right)\right]\left[exp\left(\frac{\beta_{2(v)}\left(\bar{X}-\bar{v}\right)}{\beta_{2(v)}\left(\bar{X}+\bar{v}\right)+2C_v}\right)\right]$
C_v	$\beta_{2(v)}$	$\left[\varphi_{18}\hat{\hat{\bar{Y}}}_A+\varphi_{19}\left(\bar{X}-\bar{v}\right)\right]\left[exp\left(\frac{C_v\left(\bar{X}-\bar{v}\right)}{C_v\left(\bar{X}+\bar{v}\right)+2\beta_{2(v)}}\right)\right]$
1	ρ_{uv}	$\left[\varphi_{18}\hat{\hat{\bar{Y}}}_A+\varphi_{19}\left(\bar{X}-\bar{v}\right)\right]\left[exp\left(\frac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}+\bar{v}\right)+2\rho_{uv}}\right)\right]$
C_v	ρ_{uv}	$\left[\varphi_{18}\hat{\hat{\bar{Y}}}_A+\varphi_{19}\left(\bar{X}-\bar{v}\right)\right]\left[exp\left(\frac{C_v\left(\bar{X}-\bar{v}\right)}{C_v\left(\bar{X}+\bar{v}\right)+2\rho_{uv}}\right)\right]$
ρ_{uv}	C_v	$\left[\varphi_{18}\hat{\hat{\bar{Y}}}_A+\varphi_{19}\left(\bar{X}-\bar{v}\right)\right]\left[exp\left(\frac{\rho_{uv}\left(\bar{X}-\bar{v}\right)}{\rho_{uv}\left(\bar{X}+\bar{v}\right)+2C_v}\right)\right]$
$\beta_{2(v)}$	ρ_{uv}	$\left[\varphi_{18}\hat{\hat{\bar{Y}}}_A+\varphi_{19}\left(\bar{X}-\bar{v}\right)\right]\left[exp\left(\frac{\beta_{2(v)}\left(\bar{X}-\bar{v}\right)}{\beta_{2(v)}\left(\bar{X}+\bar{v}\right)+2\rho_{uv}}\right)\right]$
ρ_{uv}	$\beta_{2(v)}$	$\left[\varphi_{18}\hat{\hat{\bar{Y}}}_A+\varphi_{19}\left(\bar{X}-\bar{v}\right)\right]\left[exp\left(\frac{\rho_{uv}\left(\bar{X}-\bar{v}\right)}{\rho_{uv}\left(\bar{X}+\bar{v}\right)+2\beta_{2(v)}}\right)\right]$
1	$N\bar{X}$	$\left[\varphi_{18}\hat{\hat{\bar{Y}}}_A+\varphi_{19}\left(\bar{X}-\bar{v}\right)\right]\left[exp\left(\frac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}+\bar{v}\right)+2N\bar{X}}\right)\right]$

$$C_z = 1+\lambda\left[\left(\frac{\{5+3\theta^2\}}{8}\right)C_v^2 - \frac{1}{2}\theta\rho_{uv}C_uC_v\right],$$

$$D_z = R\lambda\frac{\theta C_v^2}{2},$$

$$E_z = R\lambda\left[\theta C_v^2 - \rho_{uv}C_uC_v\right].$$

4. PROPOSED CLASSES OF ESTIMATORS

By utilizing the supplementary information in a proper way at any stage, whether at the design or at the estimation, the efficiency of the estimator can be enhanced. Taking clue from Sohaib *et al.* (2023), we have developed the three new classes of generalized estimators under PPS sampling .

4.1 CLASS OF ESTIMATORS - I

We formulated a generalized class of estimators for $PM\bar{Y}$ as

$$\hat{\hat{A}}_{(P1,PPS)} = \left[\alpha_1\hat{\hat{\bar{Y}}}_A + \beta_1\left(\frac{\bar{v}}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{a\bar{X}+b}{a\bar{v}+b}\right)\right\}+(1-\alpha)\left(\frac{a\bar{X}+b}{a\bar{v}+b}\right)\right], \quad (4.1)$$

where (α_1, β_1) are unknown constants and (α, a, b) are suitably chosen scalars and $\hat{\hat{\bar{Y}}}_A$ is same as defined earlier.

Expressing (4.1) in terms of (ψ_0, ψ_1) we have

$$\hat{\hat{A}}_{(P1,PPS)} = \left[\alpha_1\left(1+\psi_0+\frac{5}{8}\psi_1^2\right)+\beta_1\left(\frac{\bar{X}(1+\psi_1)}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{a\bar{X}+b}{(1+\psi_1)a\bar{X}+b}\right)\right\}+(1-\alpha)\left(\frac{a\bar{X}+b}{(1+\psi_1)a\bar{X}+b}\right)\right]$$

$$=\left[\alpha_1\left(1+\psi_0+\frac{5}{8}\psi_1^2\right)+\beta_1\left(\frac{\bar{X}(1+\psi_1)}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log(1+\theta\psi_1)^{-1}\right\}+(1+\theta\psi_1)^{-1}(1-\alpha)\right] \quad (4.2)$$

After expansion and simplification, retaining only terms up to the second order in ψ, we obtain

$$\hat{\hat{A}}_{(P1,PPS)} \cong \bar{Y}\left[\alpha_1\left\{1+\psi_0-\theta\psi_1-\theta\psi_0\psi_1+\psi_1^2\left(\theta^2\left(\frac{2-\alpha}{2}\right)+\frac{5}{8}\right)\right\}+\frac{\beta_1}{R\bar{X}}\left\{1+\psi_1(\eta-\theta)+\left\{\psi_1^2\left(\theta^2\left(\frac{2-\alpha}{2}\right)-\theta\eta+\frac{\eta(\eta-1)}{2}\right)\right\}\right\}\right],$$

or

$$\left(\hat{\hat{A}}_{(P1,PPS)}-\bar{Y}\right) \cong \bar{Y}\left[\alpha_1\left\{1+\psi_0-\theta(\psi_1+\psi_0\psi_1)+K\psi_1^2\right\}+\frac{\beta_1}{R\bar{X}}\left\{1+K^*\psi_1+\psi_1^2Q\right\}-1\right], \quad (4.3)$$

On taking the expectation of both sides of (4.2), we get the bias of $\hat{\hat{A}}_{(P1,PPS)}$ to the *foa* as

$$Bias(\hat{\hat{A}}_{(P1,PPS)}) = \bar{Y}\left[\alpha_1\left\{1+\left(C_v^2K-\theta\rho_{uv}C_uC_v\right)\lambda\right\}+\beta_1\left(\frac{1}{R\bar{X}}\right)\left\{1+\lambda QC_v^2\right\}-1\right] \quad (4.4)$$

Upon squaring both sides of (4.3) and retaining terms only up to the second order in ψ, we obtain

$$\left(\hat{\hat{A}}_{(P1,PPS)}-\bar{Y}\right)^2 = \bar{Y}^2\left[1+\alpha_1^2\left\{1+2\psi_0-2\theta\psi_1+\psi_0^2-4\theta\psi_0\psi_1+\left(\theta^2+2K\right)\psi_1^2\right\}+\left(\frac{\beta_1^2}{R^2\bar{X}^2}\right)\left\{1+2K^*\psi_1+\left(K^{*2}+2Q\right)\psi_1^2\right\}+\frac{2\alpha_1\beta_1}{R\bar{X}}\left\{1+\left(K^*-\theta\right)(\psi_1-\psi_0\psi_1)+\left(K+Q-K^*Q\right)\psi_1^2\right\}-2\alpha_1\left\{1+\psi_0-\theta(\psi_1+\psi_0\psi_1)+K\psi_1^2\right\}-\frac{2\beta_1}{R\bar{X}}\left(1+K^*\psi_1+\psi_1^2Q\right)\right] \quad (4.5)$$

Applying the expectation to both sides of (4.5), the *MSeR* of $\hat{A}_{(P1,PPS)}$ under the *foa* is obtained as

$$MSE(\hat{\hat{A}}_{(P1,PPS)}) = \bar{Y}^2\left[1+\alpha_1^2A_{1(1)}+\beta_1^2A_{2(1)}+2\alpha_1\beta_1A_{3(1)}-2\alpha_1A_{4(1)}-2\beta_1A_{5(1)}\right] \quad (4.6)$$

where $K = \theta^2\left(\frac{2-\alpha}{2}\right)+\frac{5}{8}$, $K^* = (\eta-\theta)$,

$$Q = \left(\theta^2\left(\frac{2-\alpha}{2}\right)-\theta\eta+\frac{\eta(\eta-1)}{2}\right),\ R=\frac{\bar{Y}}{\bar{X}},$$

$$A_{1(1)}=\left[1+\lambda\left\{C_u^2-4\theta\rho_{uv}C_uC_v+C_v^2\left(\frac{5}{4}+(2-\alpha)\theta^2\right)\right\}\right],$$

$$A_{2(1)}=\left[\left(\frac{1}{R^2\bar{X}^2}\right)\left\{1+\lambda C_v^2\left(K^{*2}+2\theta\right)\right\}\right],$$

$$A_{3(1)}=\left(\frac{1}{R\bar{X}}\right)\left[1+\lambda\rho_{uv}C_uC_v\left(K^*-\theta\right)+\lambda C_v^2\left(K-\theta K^*+Q\right)\right],$$

$$A_{4(1)}=\left[1+\lambda\left(C_v^2K-\theta\rho_{uv}C_uC_v\right)\right],$$

$$A_{5(1)}=\left[\left(\frac{1}{R\bar{X}}\right)\left(1+\lambda C_v^2\theta\right)\right].$$

Setting $\frac{\partial MSE(\hat{\hat{\bar{Y}}}_{(P1,PPS)})}{\partial\alpha_1}=0$ and $\frac{\partial MSE(\hat{\hat{\bar{Y}}}_{(P1,PPS)})}{\partial\beta_1}=0$, we have

$$\begin{bmatrix}A_{1(1)} & A_{3(1)}\\ A_{3(1)} & A_{2(1)}\end{bmatrix}\begin{bmatrix}\alpha_1\\ \beta_1\end{bmatrix}=\begin{bmatrix}A_{4(1)}\\ A_{5(1)}\end{bmatrix} \tag{4.7}$$

The optimum values of α_1 and β_1 can be obtained by solving (4.7) as

Table 4.1. Some members of the family of estimators $\hat{\hat{A}}_{(P1,PPS)}$

a	b	Estimators
1	C_v	$\left[\alpha_1\hat{\hat{\bar{Y}}}_A+\beta_1\left(\frac{\bar{v}}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{\bar{X}+C_v}{\bar{v}+C_v}\right)\right\}+(1-\alpha)\left(\frac{\bar{X}+C_v}{\bar{v}+C_v}\right)\right]$
1	$\beta_{2(v)}$	$\left[\alpha_1\hat{\hat{\bar{Y}}}_A+\beta_1\left(\frac{\bar{v}}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{\bar{X}+\beta_{2(v)}}{\bar{v}+\beta_{2(v)}}\right)\right\}+(1-\alpha)\left(\frac{\bar{X}+\beta_{2(v)}}{\bar{v}+\beta_{2(v)}}\right)\right]$
$\beta_{2(v)}$	C_v	$\left[\alpha_1\hat{\hat{\bar{Y}}}_A+\beta_1\left(\frac{\bar{v}}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{\beta_{2(v)}\bar{X}+C_v}{\beta_{2(v)}\bar{v}+C_v}\right)\right\}+(1-\alpha)\left(\frac{\beta_{2(v)}\bar{X}+C_v}{\beta_{2(v)}\bar{v}+C_v}\right)\right]$
C_v	$\beta_{2(v)}$	$\left[\alpha_1\hat{\hat{\bar{Y}}}_A+\beta_1\left(\frac{\bar{v}}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{C_v\bar{X}+\beta_{2(v)}}{C_v\bar{v}+\beta_{2(v)}}\right)\right\}+(1-\alpha)\left(\frac{C_v\bar{X}+\beta_{2(v)}}{C_v\bar{v}+\beta_{2(v)}}\right)\right]$
1	ρ_{uv}	$\left[\alpha_1\hat{\hat{\bar{Y}}}_A+\beta_1\left(\frac{\bar{v}}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{\bar{X}+\rho_{uv}}{\bar{v}+\rho_{uv}}\right)\right\}+(1-\alpha)\left(\frac{\bar{X}+\rho_{uv}}{\bar{v}+\rho_{uv}}\right)\right]$
C_v	ρ_{uv}	$\left[\alpha_1\hat{\hat{\bar{Y}}}_A+\beta_1\left(\frac{\bar{v}}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{C_v\bar{X}+\rho_{uv}}{C_v\bar{v}+\rho_{uv}}\right)\right\}+(1-\alpha)\left(\frac{C_v\bar{X}+\rho_{uv}}{C_v\bar{v}+\rho_{uv}}\right)\right]$
ρ_{uv}	C_v	$\left[\alpha_1\hat{\hat{\bar{Y}}}_A+\beta_1\left(\frac{\bar{v}}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{\rho_{uv}\bar{X}+C_v}{\rho_{uv}\bar{v}+C_v}\right)\right\}+(1-\alpha)\left(\frac{\rho_{uv}\bar{X}+C_v}{\rho_{uv}\bar{v}+C_v}\right)\right]$
ρ_{uv}	C_v	$\left[\alpha_1\hat{\hat{\bar{Y}}}_A+\beta_1\left(\frac{\bar{v}}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{\rho_{uv}\bar{X}+C_v}{\rho_{uv}\bar{v}+C_v}\right)\right\}+(1-\alpha)\left(\frac{\rho_{uv}\bar{X}+C_v}{\rho_{uv}\bar{v}+C_v}\right)\right]$
$\beta_{2(v)}$	ρ_{uv}	$\left[\alpha_1\hat{\hat{\bar{Y}}}_A+\beta_1\left(\frac{\bar{v}}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{\beta_{2(v)}\bar{X}+\rho_{uv}}{\beta_{2(v)}\bar{v}+\rho_{uv}}\right)\right\}+(1-\alpha)\left(\frac{\beta_{2(v)}\bar{X}+\rho_{uv}}{\beta_{2(v)}\bar{v}+\rho_{uv}}\right)\right]$
ρ_{uv}	$\beta_{2(v)}$	$\left[\alpha_1\hat{\hat{\bar{Y}}}_A+\beta_1\left(\frac{\bar{v}}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{\rho_{uv}\bar{X}+\beta_{2(v)}}{\rho_{uv}\bar{v}+\beta_{2(v)}}\right)\right\}+(1-\alpha)\left(\frac{\rho_{uv}\bar{X}+\beta_{2(v)}}{\rho_{uv}\bar{v}+\beta_{2(v)}}\right)\right]$
1	$N\bar{X}$	$\left[\alpha_1\hat{\hat{\bar{Y}}}_A+\beta_1\left(\frac{\bar{v}}{\bar{X}}\right)^\eta\right]\left[\alpha\left\{1+log\left(\frac{\bar{X}+N\bar{X}}{\bar{v}+N\bar{X}}\right)\right\}+(1-\alpha)\left(\frac{\bar{X}+N\bar{X}}{\bar{v}+N\bar{X}}\right)\right]$

$$\alpha_{1(opt)} = \frac{\Delta_{1(1)}}{\Delta_{0(1)}} \text{ and } \beta_{1(opt)} = \frac{\Delta_{2(1)}}{\Delta_{0(1)}}, \tag{4.8}$$

where

$$\Delta_{0(1)} = \begin{vmatrix} A_{1(1)} & A_{3(1)} \\ A_{3(1)} & A_{2(1)} \end{vmatrix},$$

$$= \left(A_{1(1)} A_{2(1)} - A_{3(1)}^2\right),$$

$$\Delta_{1(1)} = \begin{vmatrix} A_{4(1)} & A_{3(1)} \\ A_{5(1)} & A_{2(1)} \end{vmatrix},$$

$$= \left(A_{2(1)} A_{4(1)} - A_{3(1)} A_{5(1)}\right),$$

$$\Delta_{2(1)} = \begin{vmatrix} A_{1(1)} & A_{4(1)} \\ A_{3(1)} & A_{5(1)} \end{vmatrix},$$

$$= \left(A_{1(1)} A_{5(1)} - A_{3(1)} A_{4(1)}\right),$$

Consequently, the least attainable *MSeR* of $\hat{\hat{A}}_{(P1,PPS)}$ is given by

$$MSE_{min}\left(\hat{\hat{A}}_{(P1,PPS)}\right) = \bar{Y}^2\left[1 - M_1\right], \tag{4.9}$$

where $M_1 = \dfrac{A_{4(1)}\Delta_{1(1)} + A_{5(1)}\Delta_{2(1)}}{\Delta_{0(1)}}$.

The $MSE_{min}\left(\hat{\hat{A}}_{(P1,PPS)}\right)$ is non – negative if

$$0 < M_1 < 1 \text{ and } \left(A_{1(1)}A_{2(1)} - A_{3(1)}^2\right) > 0\,. \tag{4.10}$$

4.2 CLASS OF ESTIMATORS - II

We consider the class of generalized estimators for population mean $\bar{Y}$ as,

$$\hat{\hat{A}}_{(P2,PPS)} = \left[\alpha_2 \hat{\hat{\bar{Y}}}_A + \beta_2\left(\bar{X} - \bar{v}\right)\right]\left[\alpha\left(\frac{a\bar{X}+b}{a\bar{v}+b}\right) + (1-\alpha)\left\{1 + log\left(\frac{a\bar{X}+b}{a\bar{v}+b}\right)\right\}\right], \tag{4.11}$$

where (α_2, β_2) correspond to the constants yet to be determined such that *MSeR* of $\hat{\hat{A}}_{(P2,PPS)}$ is minimum and (α, b) are suitable chosen scalars.

Expressing (4.11) in terms of (ψ_0, ψ_1) we get

$$\hat{\hat{A}}_{(P2,PPS)} = \bar{Y}\left[\alpha_2\left(1+\psi_0+\frac{5}{8}\psi_1^2\right)+\beta_2\left(\bar{X}-\bar{X}\left(1+\psi_1\right)\right)\right] \left[\alpha\left(\frac{a\bar{X}+b}{a\bar{X}\left(1+\psi_1\right)+b}\right)+(1-\alpha)\left\{1+log\left(\frac{a\bar{X}+b}{a\bar{X}\left(1+\psi_1\right)+b}\right)\right\}\right] \tag{4.12}$$

By expanding and simplifying the right-hand side of (4.12), while retaining only second-order terms in (ψ_0, ψ_1), we obtain

$$\hat{\hat{A}}_{(P2,PPS)} \cong \bar{Y}\left[\alpha_2\left\{1+\psi_0-\theta\psi_1-\theta\psi_0\psi_1+\psi_1^2\left(\frac{5}{8}+\frac{(\alpha+1)}{2}\theta^2\right)\right\}+\frac{\beta_2}{R}\left(\theta\psi_1^2-\psi_1\right)\right]$$

or

$$\hat{\hat{A}}_{(P2,PPS)} - \bar{Y} = \bar{Y}\left[\alpha_2\left\{1+\psi_0-\theta\psi_1-\theta\psi_0\psi_1+\psi_1^2\left(\frac{5}{8}+\frac{(\alpha+1)}{2}\theta^2\right)\right\}+\frac{\beta_2}{R}\left(\theta\psi_1^2-\psi_1\right)-1\right] \tag{4.13}$$

On taking the expectation of both sides of (4.13), we obtained the bias of $\hat{\hat{A}}_{(P2,PPS)}$ to the *foa* as

$$Bias\left(\hat{\hat{A}}_{(P2,PPS)}\right) = \bar{Y}\left[\alpha_2\left\{\lambda\left[C_v^2\left(\frac{5}{8}+\frac{1}{2}(\alpha+1)\theta^2\right)-\theta\rho_{uv}C_uC_v\right]+1\right\}\beta_2\left(\frac{\lambda\theta C_v^2}{R}\right)-1\right]. \tag{4.14}$$

Squaring both sides of (4.13) and neglecting terms of (ψ_0, ψ_1) having power greater than two we have

$$\left(\hat{\hat{A}}_{(P2,PPS)} - \bar{Y}\right)^2 = \bar{Y}^2\left[1+\alpha_2^2\left\{1+2\psi_0-2\theta\psi_1+\psi_0^2-4\theta\psi_0\psi_1+\psi_1^2\left(\frac{5}{4}+(\alpha+2)\theta^2\right)\right\}+\frac{\beta_2^2}{R^2}\psi_1^2+\frac{2\alpha_2\beta_2}{R}\left(2\theta\psi_1^2-\psi_0\psi_1-\psi_1\right)-2\alpha_2\left\{1+\psi_0-\theta\psi_1-\theta\psi_0\psi_1+\psi_1^2\left(\frac{5}{8}+\frac{(\alpha+1)}{2}\theta^2\right)\right\}-2\frac{\beta_2}{R}\left(\theta\psi_1^2-\psi_1\right)\right]. \tag{4.15}$$

$$MSE\left(\hat{\hat{A}}_{(P2,PPS)}\right) = \bar{Y}^2\left[1+\alpha_2^2 A_{1(2)} + \beta_2^2 A_{2(2)} + 2\alpha_2\beta_2 A_{3(2)} - 2\alpha_2 A_{4(2)} - 2\beta_2 A_{5(2)}\right] \tag{4.16}$$

where

$$A_{1(2)} = \left[1+\lambda\left\{C_u^2 - 4\theta\rho_{uv}C_uC_v + C_v^2\left(\frac{5}{4}+(2+\alpha)\theta^2\right)\right\}\right],$$

$$A_{2(2)} = \left(\frac{\lambda}{R^2}\right)C_v^2,$$

$$A_{3(2)} = \left(\frac{\lambda}{R}\right)\left(2\theta C_v^2 - \theta\rho_{uv}C_uC_v\right),$$

$$A_{4(2)} = \left[1+\lambda\left\{\left(\frac{5}{8}+\frac{1}{2}(\alpha+1)\theta^2\right)C_v^2 - \theta\rho_{uv}C_uC_v\right\}\right],$$

$$A_{5(2)} = \left(\frac{\lambda\theta}{R^2}\right)C_v^2.$$

Proceeding as earlier, we obtain the optimum values of α_2 and β_2 as

$$\alpha_{2(opt)} = \frac{\Delta_{1(2)}}{\Delta_{0(2)}} \text{ and } \beta_{2(opt)} = \frac{\Delta_{2(2)}}{\Delta_{0(2)}}, \tag{4.17}$$

where $\Delta_{0(2)} = \left(A_{1(2)}A_{2(2)} - A_{3(2)}^2\right)$,

$\ddot{A}_{1(2)} = \left(A_{4(2)}A_{2(2)} - A_{3(2)}A_{5(2)}\right)$,

$\ddot{A}_{2(2)} = \left(A_{1(2)}A_{5(2)} - A_{3(2)}A_{4(2)}\right)$.

Thus, the minimum $MSeR$ of $\hat{\hat{A}}_{(P2,PPS)}$ given as

$$MSE_{min}\left(\hat{\hat{A}}_{(P2,PPS)}\right) = \bar{Y}^2\left[1-M_2\right], \tag{4.18}$$

where $M_2 = \dfrac{A_{4(2)}\ddot{A}_{1(2)} + A_{5(2)}\ddot{A}_{2(2)}}{\ddot{A}_{0(2)}}$.

The $MSE_{min}\left(\hat{\hat{A}}_{(P2,PPS)}\right)$ is non – negative if

$$0 < M_2 < 1 \text{ and } \left(A_{1(2)}A_{2(2)} - A_{3(2)}^2\right) > 0. \tag{4.19}$$

Table 4.2. Some members of the family of estimators $\hat{\hat{A}}_{(P2,PPS)}$

a	b	Estimators
1	C_v	$\left[\alpha_2\hat{\hat{Y}}_A + \beta_2(\bar{X}-\bar{v})\right]\left[\alpha\left(\frac{\bar{X}+C_v}{\bar{v}+C_v}\right) + (1-\alpha)\left\{1+log\left(\frac{\bar{X}+C_v}{\bar{v}+C_v}\right)\right\}\right]$
1	$\beta_{2(v)}$	$\left[\alpha_2\hat{\hat{Y}}_A + \beta_2(\bar{X}-\bar{v})\right]\left[\alpha\left(\frac{\bar{X}+\beta_{2(v)}}{\bar{v}+\beta_{2(v)}}\right) + (1-\alpha)\left\{1+log\left(\frac{\bar{X}+\beta_{2(v)}}{\bar{v}+\beta_{2(v)}}\right)\right\}\right]$
$\beta_{2(v)}$	C_v	$\left[\alpha_2\hat{\hat{Y}}_A + \beta_2(\bar{X}-\bar{v})\right]\left[\alpha\left(\frac{\beta_{2(v)}\bar{X}+C_v}{\beta_{2(v)}\bar{v}+C_v}\right) + (1-\alpha)\left\{1+log\left(\frac{\beta_{2(v)}\bar{X}+C_v}{\beta_{2(v)}\bar{v}+C_v}\right)\right\}\right]$
C_v	$\beta_{2(v)}$	$\left[\alpha_2\hat{\hat{Y}}_A + \beta_2(\bar{X}-\bar{v})\right]\left[\alpha\left(\frac{C_v\bar{X}+\beta_{2(v)}}{C_v\bar{v}+\beta_{2(v)}}\right) + (1-\alpha)\left\{1+log\left(\frac{C_v\bar{X}+\beta_{2(v)}}{C_v\bar{v}+\beta_{2(v)}}\right)\right\}\right]$
1	ρ_{uv}	$\left[\alpha_2\hat{\hat{Y}}_A + \beta_2(\bar{X}-\bar{v})\right]\left[\alpha\left(\frac{\bar{X}+\rho_{uv}}{\bar{v}+\rho_{uv}}\right) + (1-\alpha)\left\{1+log\left(\frac{\bar{X}+\rho_{uv}}{\bar{v}+\rho_{uv}}\right)\right\}\right]$
C_v	ρ_{uv}	$\left[\alpha_2\hat{\hat{Y}}_A + \beta_2(\bar{X}-\bar{v})\right]\left[\alpha\left(\frac{C_v\bar{X}+\rho_{uv}}{C_v\bar{v}+\rho_{uv}}\right) + (1-\alpha)\left\{1+log\left(\frac{C_v\bar{X}+\rho_{uv}}{C_v\bar{v}+\rho_{uv}}\right)\right\}\right]$
ρ_{uv}	C_v	$\left[\alpha_2\hat{\hat{Y}}_A + \beta_2(\bar{X}-\bar{v})\right]\left[\alpha\left(\frac{\rho_{uv}\bar{X}+C_v}{\rho_{uv}\bar{v}+C_v}\right) + (1-\alpha)\left\{1+log\left(\frac{\rho_{uv}\bar{X}+C_v}{\rho_{uv}\bar{v}+C_v}\right)\right\}\right]$
$\beta_{2(v)}$	ρ_{uv}	$\left[\alpha_2\hat{\hat{Y}}_A + \beta_2(\bar{X}-\bar{v})\right]\left[\alpha\left(\frac{\beta_{2(v)}\bar{X}+\rho_{uv}}{\beta_{2(v)}\bar{v}+\rho_{uv}}\right) + (1-\alpha)\left\{1+log\left(\frac{\beta_{2(v)}\bar{X}+\rho_{uv}}{\beta_{2(v)}\bar{v}+\rho_{uv}}\right)\right\}\right]$
ρ_{uv}	$\beta_{2(v)}$	$\left[\alpha_2\hat{\hat{Y}}_A + \beta_2(\bar{X}-\bar{v})\right]\left[\alpha\left(\frac{\rho_{uv}\bar{X}+\beta_{2(v)}}{\rho_{uv}\bar{v}+\beta_{2(v)}}\right) + (1-\alpha)\left\{1+log\left(\frac{\rho_{uv}\bar{X}+\beta_{2(v)}}{\rho_{uv}\bar{v}+\beta_{2(v)}}\right)\right\}\right]$
1	$N\bar{X}$	$\left[\alpha_2\hat{\hat{Y}}_A + \beta_2(\bar{X}-\bar{v})\right]\left[\alpha\left(\frac{\bar{X}+N\bar{X}}{\bar{v}+N\bar{X}}\right) + (1-\alpha)\left\{1+log\left(\frac{\bar{X}+N\bar{X}}{\bar{v}+N\bar{X}}\right)\right\}\right]$

4.3 CLASS OF ESTIMATORS - III

Adopting the framework of Ahmad *et al.* (2023) and the clued from Nigam and Singh (2025),

We envisaged the class of estimators under PPS sampling scheme for population mean $\bar{Y}$ as

$$\hat{\hat{A}}_{(P3,PPS)}=\left[\alpha_3\hat{\hat{\bar{Y}}}_A+\beta_3\left(\bar{X}-\bar{v}\right)\right]\left[\alpha\left\{1+log\left(\frac{a\bar{X}+b}{a\bar{v}+b}\right)\right\}+\left(1-\alpha\right)exp\left\{\frac{a\left(\bar{X}-\bar{v}\right)}{a\left(\bar{X}+\bar{v}\right)+2b}\right\}\right], \tag{4.20}$$

where (α_3,β_3) are constants to be determined such that the $MSeR$ of $\hat{\hat{A}}_{(P3,PPS)}$ is minimum and (α,b) are suitable scalars. It is interesting to note that for $\alpha=0$ and $\alpha_3=\varphi_{18}$, $\beta_3=\varphi_{19}$ in (4.20), the formulated class of estimators $\hat{\hat{A}}_{(P3,PPS)}$ reduces to the class of estimators :

$$\hat{\hat{A}}^*_{(SH,PPS)}=\left[\varphi_{18}\hat{\hat{\bar{Y}}}_A+\varphi_{19}\left(\bar{X}-\bar{v}\right)\right]\left[exp\left(\frac{a\left(\bar{X}-\bar{v}\right)}{a\left(\bar{X}+\bar{v}\right)+2b}\right)\right]$$

which is due to Ahmad *et al.* (2023).

Expressing (4.20) in terms of (ψ_0,ψ_1) we have

$$\hat{\hat{A}}_{(P3,PPS)}=\bar{Y}\left[\alpha_3\left(1+\psi_0+\frac{5}{8}\psi_1^2\right)+\beta_3\left(\bar{X}-\bar{X}\left(1+\psi_1\right)\right)\right]\left[\alpha\left\{1+log\left(\frac{a\bar{X}+b}{a\bar{X}\left(1+\psi_1\right)+b}\right)\right\}+\left(1-\alpha\right)exp\left\{\frac{a\left(\bar{X}-\bar{X}\left(1+\psi_1\right)\right)}{a\left(\bar{X}+\bar{X}\left(1+\psi_1\right)\right)+2b}\right\}\right] \tag{4.21}$$

Upon expansion of the right-hand side of (4.21), while neglecting higher-order terms in (ψ_0,ψ_1), the following expression is obtained

$$\hat{\hat{A}}_{(P3,PPS)}\cong\overline{Y\left[\alpha_3\left\{1+\psi_0-\frac{1}{2}(\alpha+1)\theta\psi_1+\frac{5}{8}\psi_1^2-\right.\right.}$$
$$\overline{\left.\left.\frac{1}{2}(\alpha+1)\theta\psi_0\psi_1\right\}-\frac{\beta_3}{R}\left\{\psi_1-\frac{1}{2}(\alpha+1)\theta\psi_1^2\right\}\right]}$$

or

$$\hat{\hat{A}}_{(P3,PPS)}-\bar{Y}\cong\bar{Y}\left[\alpha_3\left(1+\psi_0-\frac{1}{2}(\alpha+1)\theta\psi_1-\frac{1}{2}(\alpha+1)\theta\psi_0\psi_1+\frac{1}{8}\left(5+(\alpha+3)\theta^2\right)\psi_1^2\right)+\frac{\beta_3}{R}\left\{\frac{1}{2}(\alpha+1)\theta\psi_1^2-\psi_1\right\}-1\right]. \tag{4.22}$$

On taking expectation of (4.13), we obtain the bias of $\hat{\hat{A}}_{(P3,PPS)}$ to the foa as

$$Bias\left(\hat{\hat{A}}_{(P3,PPS)}\right)=\bar{Y}\left[\alpha_3\left\{1+\frac{\lambda}{8}\left\{(5+(\alpha+3)\theta^2)C_v^2-4(\alpha+1)\theta\rho_{uv}C_uC_v\right\}\right\}+\beta_3\left(\frac{\lambda}{2R}\right)(\alpha+1)\theta C_v^2\right]. \tag{4.23}$$

Squaring (4.22) and disregarding higher-order terms in (ψ_0,ψ_1), the following expression is obtained

$$\left(\hat{\hat{A}}_{(P3,PPS)}-\bar{Y}\right)^2=\bar{Y}^2\left[1+\alpha_3^2\left\{1+2\psi_0-(\alpha+1)\theta\psi_1+\psi_0^2-2(\alpha+1)\theta\psi_0\psi_1+\frac{\psi_1^2}{4}\left(5+\theta^2(\alpha^2+3\alpha+4)\right)\right\}+\frac{\beta_3^2}{R^2}\psi_1^2+\frac{2\alpha_3\beta_3}{R}\left\{(\alpha+1)\theta\psi_1^2-\psi_0\psi_1-\psi_1\right\}-2\alpha_3\left\{1+\psi_0-\frac{1}{2}(\alpha+1)\theta\psi_1+\frac{5}{8}\psi_1^2-\frac{1}{2}(\alpha+1)\theta\psi_0\psi_1+\frac{1}{8}(\alpha+3)\theta^2\psi_1^2\right\}-\frac{2\beta_3}{R}\left\{\frac{1}{2}(\alpha+1)\theta\psi_1^2-\psi_1\right\}\right. \tag{4.24}$$

On applying expectation to both sides of (4.24), we get the $MSeR$ of $\hat{\hat{A}}_{(P3,PPS)}$ to the foa as

$$MSE\left(\hat{\hat{A}}_{(P3,PPS)}\right)=\bar{Y}^2\left[1+\alpha_3^2A_{1(3)}+\beta_3^2A_{2(3)}+2\alpha_3\beta_3A_{3(3)}-2\alpha_3A_{4(3)}-2\beta_3A_{5(3)}\right] \tag{4.25}$$

where

$$A_{1(3)}=\left[1+\lambda\left\{C_u^2-2(\alpha+1)\theta\rho_{uv}C_uC_v+\frac{1}{4}\left(5+(\alpha^2+3\alpha+4)\theta^2\right)\right\}C_v^2\right],$$

Table 4.3. Some members of the family of estimators $\hat{\hat{Y}}_{(P3,PPS)}$

a	b	Estimators
1	C_v	$\left[\alpha_3\hat{\hat{Y}}_A+\beta_3\left(\bar{X}-\bar{v}\right)\right]\left[\begin{array}{l}\alpha\left\{1+log\left(\frac{\bar{X}+C_v}{\bar{v}+C_v}\right)\right\}+\\ (1-\alpha)exp\left\{\frac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}+\bar{v}\right)+2C_v}\right\}\end{array}\right]$
1	$\beta_{2(v)}$	$\left[\alpha_3\hat{\hat{Y}}_A+\beta_3\left(\bar{X}-\bar{v}\right)\right]\left[\begin{array}{l}\alpha\left\{1+log\left(\frac{\bar{X}+\beta_{2(v)}}{\bar{v}+\beta_{2(v)}}\right)\right\}+\\ (1-\alpha)exp\left\{\frac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}+\bar{v}\right)+2\beta_{2(v)}}\right\}\end{array}\right]$
$\beta_{2(v)}$	C_v	$\left[\alpha_3\hat{\hat{Y}}_A+\beta_3\left(\bar{X}-\bar{v}\right)\right]\left[\begin{array}{l}\alpha\left\{1+log\left(\frac{\beta_{2(v)}\bar{X}+C_v}{\beta_{2(v)}\bar{v}+C_v}\right)\right\}+\\ (1-\alpha)exp\left\{\frac{\beta_{2(v)}\left(\bar{X}-\bar{v}\right)}{\beta_{2(v)}\left(\bar{X}+\bar{v}\right)+2C_v}\right\}\end{array}\right]$
C_v	$\beta_{2(v)}$	$\left[\alpha_3\hat{\hat{Y}}_A+\beta_3\left(\bar{X}-\bar{v}\right)\right]\left[\begin{array}{l}\alpha\left\{1+log\left(\frac{C_v\bar{X}+\beta_{2(v)}}{C_v\bar{v}+\beta_{2(v)}}\right)\right\}+\\ (1-\alpha)exp\left\{\frac{C_v\left(\bar{X}-\bar{v}\right)}{C_v\left(\bar{X}+\bar{v}\right)+2\beta_{2(v)}}\right\}\end{array}\right]$
1	ρ_{uv}	$\left[\alpha_3\hat{\hat{Y}}_A+\beta_3\left(\bar{X}-\bar{v}\right)\right]\left[\begin{array}{l}\alpha\left\{1+log\left(\frac{\bar{X}+\rho_{uv}}{\bar{v}+\rho_{uv}}\right)\right\}+\\ (1-\alpha)exp\left\{\frac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}+\bar{v}\right)+2\rho_{uv}}\right\}\end{array}\right]$
C_v	ρ_{uv}	$\left[\alpha_3\hat{\hat{Y}}_A+\beta_3\left(\bar{X}-\bar{v}\right)\right]\left[\begin{array}{l}\alpha\left\{1+log\left(\frac{C_v\bar{X}+\rho_{uv}}{C_v\bar{v}+\rho_{uv}}\right)\right\}+\\ (1-\alpha)exp\left\{\frac{C_v\left(\bar{X}-\bar{v}\right)}{C_v\left(\bar{X}+\bar{v}\right)+2\rho_{uv}}\right\}\end{array}\right]$
ρ_{uv}	C_v	$\left[\alpha_3\hat{\hat{Y}}_A+\beta_3\left(\bar{X}-\bar{v}\right)\right]\left[\begin{array}{l}\alpha\left\{1+log\left(\frac{\rho_{uv}\bar{X}+C_v}{\rho_{uv}\bar{v}+C_v}\right)\right\}+\\ (1-\alpha)exp\left\{\frac{\rho_{uv}\left(\bar{X}-\bar{v}\right)}{\rho_{uv}\left(\bar{X}+\bar{v}\right)+2C_v}\right\}\end{array}\right]$
$\beta_{2(v)}$	ρ_{uv}	$\left[\alpha_3\hat{\hat{Y}}_A+\beta_3\left(\bar{X}-\bar{v}\right)\right]\left[\begin{array}{l}\alpha\left\{1+log\left(\frac{\beta_{2(v)}\bar{X}+\rho_{uv}}{\beta_{2(v)}\bar{v}+\rho_{uv}}\right)\right\}+\\ (1-\alpha)exp\left\{\frac{\beta_{2(v)}\left(\bar{X}-\bar{v}\right)}{\beta_{2(v)}\left(\bar{X}+\bar{v}\right)+2\rho_{uv}}\right\}\end{array}\right]$
ρ_{uv}	$\beta_{2(v)}$	$\left[\alpha_3\hat{\hat{Y}}_A+\beta_3\left(\bar{X}-\bar{v}\right)\right]\left[\begin{array}{l}\alpha\left\{1+log\left(\frac{\rho_{uv}\bar{X}+\beta_{2(v)}}{\rho_{uv}\bar{v}+\beta_{2(v)}}\right)\right\}+\\ (1-\alpha)exp\left\{\frac{\rho_{uv}\left(\bar{X}-\bar{v}\right)}{\rho_{uv}\left(\bar{X}+\bar{v}\right)+2\beta_{2(v)}}\right\}\end{array}\right]$
1	$N\bar{X}$	$\left[\alpha_3\hat{\hat{Y}}_A+\beta_3\left(\bar{X}-\bar{v}\right)\right]\left[\begin{array}{l}\alpha\left\{1+log\left(\frac{\bar{X}+N\bar{X}}{\bar{v}+N\bar{X}}\right)\right\}+\\ (1-\alpha)exp\left\{\frac{\left(\bar{X}-\bar{v}\right)}{\left(\bar{X}+\bar{v}\right)+2N\bar{X}}\right\}\end{array}\right]$

$$A_{2(3)}=\left(\frac{\lambda}{R^2}\right)C_v^2,$$

$$A_{3(3)}=\left(\frac{\lambda}{R}\right)\left\{(\alpha+1)\theta C_v^2-\rho_{uv}C_uC_v\right\},$$

$$A_{4(3)}=\left[1+\frac{\lambda}{8}\left\{(5+(\alpha+3)\theta^2)C_v^2-4(\alpha+1)\theta\rho_{uv}C_uC_v\right\}\right],$$

$$A_{5(3)}=\left(\frac{\lambda}{2R}\right)(\alpha+1)\theta C_v^2.$$

The optimum values of α_3 and β_3 are respectively given by

$$\alpha_{3(opt)}=\frac{\Delta_{1(3)}}{\Delta_{0(3)}}\text{ and }\beta_{3(opt)}=\frac{\Delta_{2(3)}}{\Delta_{0(3)}}, \tag{4.26}$$

where $\Delta_{0(3)}=\left(A_{1(3)}A_{2(3)}-A_{3(3)}^2\right)$,

$\Delta_{1(3)}=\left(A_{4(3)}A_{2(3)}-A_{3(3)}A_{5(3)}\right)$,

$\Delta_{2(3)}=\left(A_{1(3)}A_{5(3)}-A_{3(3)}A_{4(3)}\right)$,

Thus, the resulting minimum MSE of $\hat{\hat{A}}_{(P3,PPS)}$ is given by

$$MSE_{min}\left(\hat{\hat{A}}_{(P3,PPS)}\right)=\bar{Y}^2\left[1-M_3\right], \tag{4.27}$$

where $M_3=\dfrac{A_{4(3)}\Delta_{1(3)}+A_{5(3)}\Delta_{2(3)}}{\Delta_{0(3)}}$,

The $MSE_{min}\left(\hat{\hat{A}}_{(P3,PPS)}\right)$ is non – negative if

$0<M_3<1$ and $\left(A_{1(3)}A_{2(3)}-A_{3(3)}^2\right)>0$.(4.28)

5. CONDITIONS FOR THE EFFICACY OF PROPOSED CLASSES OF ESTIMATORS

In this section, the minimum *MSeR* of the three proposed estimator classes are compared with those of the estimators reviewed in Section 3, and the conditions under which the proposed estimators exhibit superior performance are established.

From (3.2), (3.4), (3.6),(3.8), (3.10), (3.13), (3.14), (3.16), (3.18), (3.20), (3.22), (3.24), (3.26), (4.6), (4.18) and (4.27) we note that,

1) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right)<Var\left(\hat{\hat{A}}_{(u,PPS)}\right)$ if

$$\lambda C_u^2 + M_j > 1, \tag{5.1}$$

j = 1, 2, 3 ;

2) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right) < MSE\left(\hat{\hat{A}}_{(r,PPS)}\right)$ if

$$\lambda\left[C_u^2 + C_v^2 - 2\rho_{uv}C_uC_v\right] + M_j > 1, \tag{5.2}$$

j = 1, 2, 3;

3) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right) < MSE\left(\hat{\hat{A}}_{(p,PPS)}\right)$ if

$$\lambda\left[C_u^2 + C_v^2 + 2\rho_{uv}C_uC_v\right] + M_j > 1, \tag{5.3}$$

j = 1, 2, 3;

4) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right) < MSE_{min}\left(\hat{\hat{A}}_{(reg,PPS)}\right)$ if

$$C_u^2\lambda\left(1-\rho_{uv}^2\right) + M_j > 1, \tag{5.4}$$

j = 1, 2, 3;

5) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right) < MSE_{min}\left(\hat{\hat{A}}_{(Bai,PPS)}\right)$ if

$$\frac{\lambda C_u^2\left(1-\rho_{uv}^2\right)}{\left\{1+\lambda C_u^2\left(1-\rho_{uv}^2\right)\right\}} + M_j > 1, \tag{5.5}$$

j = 1, 2, 3;

6) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right) < MSE(\hat{\hat{A}}_{(BTr,PPS)})$ if

$$\lambda\left(C_u^2 + \frac{1}{4}C_v^2 - \rho_{uv}C_uC_v\right) + M_j > 1, \tag{5.6}$$

j = 1, 2, 3;

7) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right) < MSE(\hat{\hat{A}}_{(BTp,PPS)})$ if

$$\lambda\left(C_u^2 + \frac{1}{4}C_v^2 + \rho_{uv}C_uC_v\right) + M_j > 1, \tag{5.7}$$

j = 1, 2, 3;

8) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right) < MSE_{min}\left(\hat{\hat{A}}_{(HS1,PPS)}\right)$ if

$$M_j > \left(\frac{A_{H1}D_{H1}^2 + B_{H1}C_{H1}^2 - C_{H1}D_{H1}E_{H1}}{A_{H1}B_{H1} - E_{H1}^2}\right), \tag{5.8}$$

j = 1, 2, 3;

9) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right) < MSE_{min}\left(\hat{\hat{A}}_{(HS1,PPS)}\right)$ if

$$M_j > \left(\frac{A_{H2}D_{H2}^2 + B_{H2}C_{H2}^2 - 2C_{H2}D_{H2}E_{H2}}{A_{H2}B_{H2} - E_{H2}^2}\right), \tag{5.9}$$

j = 1, 2, 3;

10) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right) < MSE_{min}\left(\hat{\hat{A}}_{(E,PPS)}\right)$ if

$$M_j > \left(\frac{A_E D_E^2 + B_E C_E^2 - 2C_E D_E E_E}{A_E B_E - E_E^2}\right), \tag{5.10}$$

j = 1, 2, 3

11) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right) < MSE_{min}(\hat{\hat{A}}^*_{(S,PPS)})$ if

$$\frac{\lambda}{4}\left[4C_u^2 + \theta^2 C_v^2 - 4\theta\rho_{uv}C_uC_v\right] + M_j > 1, \tag{5.11}$$

j = 1, 2, 3;

12) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right) < MSE_{min}(\hat{\hat{A}}^*_{(G,PPS)})$ if

$$M_j > \left(\frac{A_k D_k^2 + B_k C_k^2 - 2C_k D_k E_k}{A_k B_k - E_k^2}\right), \tag{5.12}$$

j = 1, 2, 3;

13) $MSE_{min}\left(\hat{\hat{A}}_{(Pj,PPS)}\right) < MSE_{min}(\hat{\hat{A}}^*_{(SH,PPS)})$ if

$$M_1 > \left(\frac{A_z D_z^2 + B_z C_z^2 - 2C_z D_z E_z}{A_z B_z - E_z^2}\right), \tag{5.13}$$

j = 1, 2, 3.

Thus, the proposed class of estimators $\hat{\hat{A}}_{(Pj,PPS)}$, j = 1, 2, 3; is more efficient than the estimators $\hat{\hat{A}}_{(U,PPS)}$, $\hat{\hat{A}}_{(r,PPS)}$, $\hat{\hat{A}}_{(p,PPS)}$, $\hat{\hat{A}}_{(reg,PPS)}$, $\hat{\hat{A}}_{(Bai,PPS)}$, $\hat{\hat{A}}_{(BTr,PPS)}$, $\hat{\hat{Y}}_{(BTp,PPS)}$, $\hat{\hat{Y}}_{(HS1,PPS)}$, $\hat{\hat{Y}}_{(HS2,PPS)}$, $\hat{\hat{Y}}_{(E,PPS)}$, $\hat{\hat{Y}}^*_{(S,PPS)}$, $\hat{\hat{Y}}^*_{(G,PPS)}$ and $\hat{\hat{Y}}^*_{(SH,PPS)}$ as long as the conditions (5.1) – (5.13) are respectively satisfied.

6. EMPIRICAL EVALUATION

To evaluate the performance of the proposed classes of estimators with respect to the other estimators which are taken into consideration. We have computed the PREs of $\hat{\hat{Y}}_{(P1,PPS)}$, $\hat{\hat{Y}}_{(P2,PPS)}$ and $\hat{\hat{Y}}_{(P3,PPS)}$ with respect to $\hat{\hat{Y}}_{(u,PPS)}$ at different values of the scalars (a, b) and also for the values of the constants (η, α) and compare it with the other estimators. The PRE of the estimators is calculated utilizing the formula:

$$\text{PRE} = \left[\frac{Var\left(\hat{\hat{Y}}_{u,PPS}\right)}{\xi}\right] \text{x } 100$$

where $\xi = MSE\left(\hat{\hat{A}}_{(U,PPS)}\right)$,

$MSE\left(\hat{\hat{A}}_{(r,PPS)}\right) MSE\left(\hat{\hat{A}}_{(p,PPS)}\right)$, $MSE_{min}\left(\hat{\hat{A}}_{(reg,PPS)}\right)$,

$MSE_{min}\left(\hat{\hat{A}}_{(Bai,PPS)}\right)$, $MSE(\hat{\hat{A}}_{(BTr,PPS)})$, $MSE(\hat{\hat{A}}_{(BTp,PPS)})$

, $MSE_{min}\left(\hat{\hat{A}}_{(HS1,PPS)}\right)$, $MSE_{min}\left(\hat{\hat{A}}_{(HS2,PPS)}\right)$,

$MSE_{min}\left(\hat{\hat{A}}_{(E,PPS)}\right)$, $MSE_{min}(\hat{\hat{A}}^{*}_{(S,PPS)})$, $MSE_{min}(\hat{\hat{A}}^{*}_{(G,PPS)})$

, $MSE_{min}(\hat{\hat{A}}^{*}_{(SH,PPS)})$, $MSE_{min}\left(\hat{\hat{A}}_{(P1,PPS)}\right)$,

$MSE_{min}\left(\hat{\hat{A}}_{(P2,PPS)}\right)$, $MSE_{min}\left(\hat{\hat{A}}_{(P3,PPS)}\right)$.

6.1 POPULATION I : [Source : Punjab Bureau of Statistics (2021 – 22)]

Y : Total number of beds on the 30th June 2021.

X : Total number of beds allocated for COVID-19, 2021.

Z : Beds used by COVID-19, 2021

6.2 POPULATION II: [Source :Singh (2003)]

Y : Expected total of fist in the year 1995.

X : Expected total of fish in the year 1994.

Z : Expected total of fish in the year 1992.

Table 6.1. Summary statistics of real data set

Parameters	Population - I	Population – II
N	36	69
n	7	14
$\bar{Y}$	76.22889	4514.899
$\bar{X}$	14.77222	4954.435
C_u	1.568599	0.8523346
C_v	0.7550356	0.8925777
ρ_{uv}	0.3004429	0.1542462
C_{uv}	0.3558289	0.1173466
$\beta_{2(v)}$	2.567602	9.985055

Table 6.2. PRE of estimators with respect to $\hat{\hat{A}}_{(U,PPS)}$ at different values of the constants for population - I

α	η	PRE							
0	-1	Estimator		$\hat{\hat{A}}^{*}_{(S,PPS)}$	$\hat{\hat{A}}^{*}_{(G,PPS)}$	$\hat{\hat{A}}^{*}_{(SH,PPS)}$	$\hat{\hat{A}}_{(P1,PPS)}$	$\hat{\hat{A}}_{(P2,PPS)}$	$\hat{\hat{A}}_{(P3,PPS)}$
		$\hat{\hat{A}}_{(U,PPS)}$	100	109.31	153.90	161.05	209.38	176.51	161.05
		$\hat{\hat{A}}_{(r,PPS)}$	106.10	108.83	149.93	160.33	218.52	165.39	160.33
		$\hat{\hat{A}}_{(p,PPS)}$	65.74	109.42	155.26	161.27	208.16	180.52	161.27
		$\hat{\hat{A}}_{(reg,PPS)}$	109.92	108.61	148.63	160.07	223.60	161.91	160.07
		$\hat{\hat{A}}_{(Bai,PPS)}$	145.92	109.42	155.22	161.27	208.17	183.07	161.27
		$\hat{\hat{A}}_{(BTr,PPS)}$	109.49	109.39	154.94	161.22	208.36	182.75	161.22
		$\hat{\hat{A}}_{(BTp,PPS)}$	83.16	108.84	150.03	160.325	218.20	180.75	160.35
		$\hat{\hat{A}}_{(HS1,PPS)}$	158.53	109.46	155.83	161.36	207.93	183.29	161.36
		$\hat{\hat{A}}_{(HS2,PPS)}$	150.57	107.33	144.26	159.06	255.12	182.65	159.06
		$\hat{\hat{A}}_{(E,PPS)}$	148.99	*	*	*	*	*	*

0	0	$\hat{A}_{(U,PPS)}$	100	109.31	153.90	161.05	735.78	176.51	161.05
		$\hat{A}_{(r,PPS)}$	106.10	108.83	149.93	160.33	924.02	165.39	160.33
		$\hat{A}_{(p,PPS)}$	65.74	109.42	155.26	161.27	692.20	180.52	161.27
		$\hat{A}_{(reg,PPS)}$	109.92	108.61	148.63	160.07	1019.78	161.91	160.07
		$\hat{A}_{(Bai,PPS)}$	145.92	109.42	155.22	161.27	629.79	183.07	161.27
		$\hat{A}_{(BTr,PPS)}$	109.49	109.39	154.94	161.22	701.87	182.75	161.22
		$\hat{A}_{(BTp,PPS)}$	83.16	108.84	150.03	160.325	917.96	180.75	160.35
		$\hat{A}_{(HS1,PPS)}$	158.53	109.46	155.83	161.36	675.92	183.29	161.36
		$\hat{A}_{(HS2,PPS)}$	150.57	107.33	144.26	159.06	1747.38	182.65	159.06
		$\hat{A}_{(E,PPS)}$	148.99	*	*	*	*	*	*
0	1	$\hat{A}_{(U,PPS)}$	100	109.31	153.90	161.05	194.17	176.51	161.05
		$\hat{A}_{(r,PPS)}$	106.10	108.83	149.93	160.33	194.75	165.39	160.33
		$\hat{A}_{(p,PPS)}$	65.74	109.42	155.26	161.27	193.91	180.52	161.27
		$\hat{A}_{(reg,PPS)}$	109.92	108.61	148.63	160.07	194.85	161.91	160.07
		$\hat{A}_{(Bai,PPS)}$	145.92	109.42	155.22	161.27	193.92	183.07	161.27
		$\hat{A}_{(BTr,PPS)}$	109.49	109.39	154.94	161.22	193.98	182.75	161.22
		$\hat{A}_{(BTp,PPS)}$	83.16	108.84	150.03	160.325	194.74	180.75	160.35
		$\hat{A}_{(HS1,PPS)}$	158.53	109.46	155.83	161.36	193.80	183.29	161.36
		$\hat{A}_{(HS2,PPS)}$	150.57	107.33	144.26	159.06	194.51	182.65	159.06
		$\hat{A}_{(E,PPS)}$	148.99	*	*	*	*	*	*
0.5	-1	$\hat{A}_{(U,PPS)}$	100	109.31	153.90	161.05	210.58	165.37	325.08
		$\hat{A}_{(r,PPS)}$	106.10	108.83	149.93	160.33	221.32	131.51	323.66
		$\hat{A}_{(p,PPS)}$	65.74	109.42	155.26	161.27	208.35	173.05	325.52
		$\hat{A}_{(reg,PPS)}$	109.92	108.61	148.63	160.07	226.64	112.27	323.15
		$\hat{A}_{(Bai,PPS)}$	145.92	109.42	155.22	161.27	208.37	175.40	324.95
		$\hat{A}_{(BTr,PPS)}$	109.49	109.39	154.94	161.22	208.82	171.77	324.69
		$\hat{A}_{(BTp,PPS)}$	83.16	108.84	150.03	160.35	220.97	117.27	325.00
		$\hat{A}_{(HS1,PPS)}$	158.53	109.46	155.83	161.36	207.59	177.36	325.47
		$\hat{A}_{(HS2,PPS)}$	150.57	107.33	144.26	159.06	258.29	172.25	323.48
		$\hat{A}_{(E,PPS)}$	148.99	*	*	*	*	*	*

0.5	0	$\hat{A}_{(U,PPS)}$	100	109.31	153.90	161.05	144.46	165.37	325.08
		$\hat{A}_{(r,PPS)}$	106.10	108.83	149.93	160.33	159.84	131.51	323.66
		$\hat{A}_{(p,PPS)}$	65.74	109.42	155.26	161.27	140.40	173.05	325.52
		$\hat{A}_{(reg,PPS)}$	109.92	108.61	148.63	160.07	166.50	112.27	323.15
		$\hat{A}_{(Bai,PPS)}$	145.92	109.42	155.22	161.27	140.46	175.40	324.95
		$\hat{A}_{(BTr,PPS)}$	109.49	109.39	154.94	161.22	141.34	171.77	324.69
		$\hat{A}_{(BTp,PPS)}$	83.16	108.84	150.03	160.35	159.39	117.27	325.00
		$\hat{A}_{(HS1,PPS)}$	158.53	109.46	155.83	161.36	138.79	177.36	325.47
		$\hat{A}_{(HS2,PPS)}$	150.57	107.33	144.26	159.06	204.51	172.25	323.48
		$\hat{A}_{(E,PPS)}$	148.99	*	*	*	*	*	*
0.5	1	$\hat{A}_{(U,PPS)}$	100	109.31	153.90	161.05	178.43	165.37	325.08
		$\hat{A}_{(r,PPS)}$	106.10	108.83	149.93	160.33	181.77	131.51	323.66
		$\hat{A}_{(p,PPS)}$	65.74	109.42	155.26	161.27	177.31	173.05	325.52
		$\hat{A}_{(reg,PPS)}$	109.92	108.61	148.63	160.07	182.89	112.27	323.15
		$\hat{A}_{(Bai,PPS)}$	145.92	109.42	155.22	161.27	177.33	175.40	324.95
		$\hat{A}_{(BTr,PPS)}$	109.49	109.39	154.94	161.22	177.57	171.77	324.69
		$\hat{A}_{(BTp,PPS)}$	83.16	108.84	150.03	160.35	181.69	117.27	325.00
		$\hat{A}_{(HS1,PPS)}$	158.53	109.46	155.83	161.36	176.84	177.36	325.47
		$\hat{A}_{(HS2,PPS)}$	150.57	107.33	144.26	159.06	186.68	172.25	323.48
		$\hat{A}_{(E,PPS)}$	148.99	*	*	*	*	*	*
1	-1	$\hat{A}_{(U,PPS)}$	100	109.31	153.90	161.05	212.13	168.27	349.51
		$\hat{A}_{(r,PPS)}$	106.10	108.83	149.93	160.33	224.55	133.17	343.74
		$\hat{A}_{(p,PPS)}$	65.74	109.42	155.26	161.27	210.09	176.25	351.27
		$\hat{A}_{(reg,PPS)}$	109.92	108.61	148.63	160.07	230.02	113.26	341.58
		$\hat{A}_{(Bai,PPS)}$	145.92	109.42	155.22	161.27	210.13	178.69	348.99
		$\hat{A}_{(BTr,PPS)}$	109.49	109.39	154.94	161.22	210.70	174.91	347.96
		$\hat{A}_{(BTp,PPS)}$	83.16	108.84	150.03	160.35	224.19	118.44	349.21
		$\hat{A}_{(HS1,PPS)}$	158.53	109.46	155.83	161.36	209.08	180.72	351.05
		$\hat{A}_{(HS2,PPS)}$	150.57	107.33	144.26	159.06	261.62	176.38	342.97
		$\hat{A}_{(E,PPS)}$	148.99	*	*	*	*	*	*

1	0	$\hat{\hat{A}}_{(U,PPS)}$	100	109.31	153.90	161.05	888.66	168.27	349.51
		$\hat{\hat{A}}_{(r,PPS)}$	106.10	108.83	149.93	160.33	1117.52	133.17	343.74
		$\hat{\hat{A}}_{(p,PPS)}$	65.74	109.42	155.26	161.27	835.60	176.25	351.27
		$\hat{\hat{A}}_{(reg,PPS)}$	109.92	108.61	148.63	160.07	1233.69	113.26	341.58
		$\hat{\hat{A}}_{(Bai,PPS)}$	145.92	109.42	155.22	161.27	836.32	178.69	348.99
		$\hat{\hat{A}}_{(BTr,PPS)}$	109.49	109.39	154.94	161.22	847.37	174.91	347.96
		$\hat{\hat{A}}_{(BTp,PPS)}$	83.16	108.84	150.03	160.35	1110.16	118.44	349.21
		$\hat{\hat{A}}_{(HS1,PPS)}$	158.53	109.46	155.83	161.36	815.76	180.72	351.05
		$\hat{\hat{A}}_{(HS2,PPS)}$	150.57	107.33	144.26	159.06	1455.36	176.38	342.97
		$\hat{\hat{A}}_{(E,PPS)}$	148.99	*	*	*	*	*	*
1	1	$\hat{\hat{A}}_{(U,PPS)}$	100	109.31	153.90	161.05	212.86	168.27	349.51
		$\hat{\hat{A}}_{(r,PPS)}$	106.10	108.83	149.93	160.33	224.55	133.17	343.74
		$\hat{\hat{A}}_{(p,PPS)}$	65.74	109.42	155.26	161.27	210.09	176.25	351.27
		$\hat{\hat{A}}_{(reg,PPS)}$	109.92	108.61	148.63	160.07	230.02	113.26	341.58
		$\hat{\hat{A}}_{(Bai,PPS)}$	145.92	109.42	155.22	161.27	210.13	178.69	348.99
		$\hat{\hat{A}}_{(BTr,PPS)}$	109.49	109.39	154.94	161.22	210.70	174.91	347.96
		$\hat{\hat{A}}_{(BTp,PPS)}$	83.16	108.84	150.03	160.35	224.19	118.44	349.21
		$\hat{\hat{A}}_{(HS1,PPS)}$	158.53	109.46	155.83	161.36	209.08	180.72	351.05
		$\hat{\hat{A}}_{(HS2,PPS)}$	150.57	107.33	144.26	159.06	261.62	176.38	342.97
		$\hat{\hat{A}}_{(E,PPS)}$	148.99	*	*	*	*	*	*

Table 6.2. PRE of estimators with respect to $\hat{\hat{A}}_{(U,PPS)}$ at different values of the constants for population – II

α	η	PRE							
0	-1	**Estimator**		$\hat{\hat{A}}^{*}_{(S,\,PPS)}$	$\hat{\hat{A}}^{*}_{(K,\,PPS)}$	$\hat{\hat{A}}^{*}_{(SH,\,PPS)}$	$\hat{\hat{A}}_{(P1,PPS)}$	$\hat{\hat{A}}_{(P2,PPS)}$	$\hat{\hat{A}}_{(P3,PPS)}$
		$\hat{\hat{A}}_{(U,PPS)}$	100	89.88	116.93	120.98	129.38	126.37	120.98
		$\hat{\hat{A}}_{(r,PPS)}$	56.38	89.94	116.74	120.97	132.76	126.41	120.97
		$\hat{\hat{A}}_{(p,PPS)}$	41.32	89.87	116.95	120.98	129.09	126.35	120.98
		$\hat{\hat{A}}_{(reg,PPS)}$	102.43	89.94	116.71	120.97	133.22	126.42	120.97
		$\hat{\hat{A}}_{(Bai,PPS)}$	107.62	89.87	116.95	120.98	129.11	126.36	120.98

		$\hat{\hat{A}}_{(BTr,PPS)}$	89.87	89.87	116.95	120.98	129.12	126.36	120.98
		$\hat{\hat{A}}_{(BTp,PPS)}$	69.65	89.91	116.83	120.97	131.18	126.39	120.97
		$\hat{\hat{A}}_{(HS1,PPS)}$	118.08	89.87	116.95	120.98	129.06	126.35	120.98
		$\hat{\hat{A}}_{(HS2,PPS)}$	111.10	90.27	115.62	120.89	157.64	126.48	120.89
		$\hat{\hat{A}}_{(E,PPS)}$	111.12	*	*	*	*	*	*
0	0	$\hat{\hat{A}}_{(U,PPS)}$	100	89.88	116.93	120.98	389.69	126.37	120.98
		$\hat{\hat{A}}_{(r,PPS)}$	56.38	89.94	116.74	120.97	391.20	126.41	120.97
		$\hat{\hat{A}}_{(p,PPS)}$	41.32	89.87	116.95	120.98	389.54	126.35	120.98
		$\hat{\hat{A}}_{(reg,PPS)}$	102.43	89.94	116.71	120.97	393.23	126.42	120.97
		$\hat{\hat{A}}_{(Bai,PPS)}$	107.62	89.87	116.95	120.98	389.55	126.36	120.98
		$\hat{\hat{A}}_{(BTr,PPS)}$	89.87	89.87	116.95	120.98	389.59	126.36	120.98
		$\hat{\hat{A}}_{(BTp,PPS)}$	69.65	89.91	116.83	120.97	390.02	126.39	120.97
		$\hat{\hat{A}}_{(HS1,PPS)}$	118.08	89.87	116.95	120.98	389.53	126.35	120.98
		$\hat{\hat{A}}_{(HS2,PPS)}$	111.10	90.27	115.62	120.89	400.44	126.48	120.89
		$\hat{\hat{A}}_{(E,PPS)}$	111.12	*	*	*	*	*	*
0	1	$\hat{\hat{A}}_{(U,PPS)}$	100	89.88	116.93	120.98	141.48	126.37	120.98
		$\hat{\hat{A}}_{(r,PPS)}$	56.38	89.94	116.74	120.97	142.09	126.41	120.97
		$\hat{\hat{A}}_{(p,PPS)}$	41.32	89.87	116.95	120.98	141.47	126.35	120.98
		$\hat{\hat{A}}_{(reg,PPS)}$	102.43	89.94	116.71	120.97	145.60	126.42	120.97
		$\hat{\hat{Y}}_{(Bai,PPS)}$	107.62	89.87	116.95	120.98	141.50	126.36	120.98
		$\hat{\hat{A}}_{(BTr,PPS)}$	89.87	89.87	116.95	120.98	141.47	126.36	120.98
		$\hat{\hat{A}}_{(BTp,PPS)}$	69.65	89.91	116.83	120.97	141.49	126.39	120.97
		$\hat{\hat{A}}_{(HS1,PPS)}$	118.08	89.87	116.95	120.98	141.47	126.35	120.98
		$\hat{\hat{A}}_{(HS2,PPS)}$	111.10	90.27	115.62	120.89	142.59	126.48	120.89
		$\hat{\hat{A}}_{(E,PPS)}$	111.12	*	*	*	*	*	*

0.5	-1	$\hat{A}_{(U,PPS)}$	100	89.88	116.93	120.98	515.33	131.57	567.36
		$\hat{A}_{(r,PPS)}$	56.38	89.94	116.74	120.97	516.35	131.61	568.40
		$\hat{A}_{(p,PPS)}$	41.32	89.87	116.95	120.98	515.31	131.55	562.39
		$\hat{A}_{(reg,PPS)}$	102.43	89.94	116.71	120.97	520.70	131.62	562.65
		$\hat{A}_{(Bai,PPS)}$	107.62	89.87	116.95	120.98	515.33	131.56	562.78
		$\hat{A}_{(BTr,PPS)}$	89.87	89.87	116.95	120.98	515.39	131.56	563.80
		$\hat{A}_{(BTp,PPS)}$	69.65	89.91	116.83	120.97	515.44	131.59	536.82
		$\hat{A}_{(HS1,PPS)}$	118.08	89.87	116.95	120.98	515.31	131.55	538.65
		$\hat{A}_{(HS2,PPS)}$	111.10	90.27	115.62	120.89	522.04	131.68	560.51
		$\hat{A}_{(E,PPS)}$	111.12	*	*	*	*	*	*
0.5	0	$\hat{A}_{(U,PPS)}$	100	89.88	116.93	120.98	535.41	131.57	567.36
		$\hat{A}_{(r,PPS)}$	56.38	89.94	116.74	120.97	537.61	131.61	568.40
		$\hat{A}_{(p,PPS)}$	41.32	89.87	116.95	120.98	535.33	131.55	562.39
		$\hat{A}_{(reg,PPS)}$	102.43	89.94	116.71	120.97	541.17	131.62	562.65
		$\hat{A}_{(Bai,PPS)}$	107.62	89.87	116.95	120.98	535.36	131.56	562.78
		$\hat{A}_{(BTr,PPS)}$	89.87	89.87	116.95	120.98	535.41	131.56	563.80
		$\hat{A}_{(BTp,PPS)}$	69.65	89.91	116.83	120.97	535.91	131.59	536.82
		$\hat{A}_{(HS1,PPS)}$	118.08	89.87	116.95	120.98	535.33	131.55	538.65
		$\hat{A}_{(HS2,PPS)}$	111.10	90.27	115.62	120.89	550.28	131.68	560.51
		$\hat{A}_{(E,PPS)}$	111.12	*	*	*	*	*	*
0.5	1	$\hat{A}_{(U,PPS)}$	100	89.88	116.93	120.98	532.53	131.57	567.36
		$\hat{A}_{(r,PPS)}$	56.38	89.94	116.74	120.97	534.33	131.61	568.40
		$\hat{A}_{(p,PPS)}$	41.32	89.87	116.95	120.98	532.51	131.55	562.39
		$\hat{A}_{(reg,PPS)}$	102.43	89.94	116.71	120.97	552.50	131.62	562.65
		$\hat{A}_{(Bai,PPS)}$	107.62	89.87	116.95	120.98	532.55	131.56	562.78
		$\hat{A}_{(BTr,PPS)}$	89.87	89.87	116.95	120.98	533.01	131.56	563.80
		$\hat{A}_{(BTp,PPS)}$	69.65	89.91	116.83	120.97	532.58	131.59	536.82
		$\hat{A}_{(HS1,PPS)}$	118.08	89.87	116.95	120.98	532.51	131.55	538.65
		$\hat{A}_{(HS2,PPS)}$	111.10	90.27	115.62	120.89	539.71	131.68	560.51
		$\hat{A}_{(E,PPS)}$	111.12	*	*	*	*	*	*

1	-1	$\hat{A}_{(U,PPS)}$	100	89.88	116.93	120.98	117.28	168.27	282.65
		$\hat{A}_{(r,PPS)}$	56.38	89.94	116.74	120.97	117.40	133.17	282.80
		$\hat{A}_{(p,PPS)}$	41.32	89.87	116.95	120.98	117.28	176.25	280.60
		$\hat{A}_{(reg,PPS)}$	102.43	89.94	116.71	120.97	118.57	113.26	283.50
		$\hat{A}_{(Bai,PPS)}$	107.62	89.87	116.95	120.98	117.29	178.69	286.36
		$\hat{A}_{(BTr,PPS)}$	89.87	89.87	116.95	120.98	117.32	174.91	288.58
		$\hat{A}_{(BTp,PPS)}$	69.65	89.91	116.83	120.97	117.29	118.44	282.80
		$\hat{A}_{(HS1,PPS)}$	118.08	89.87	116.95	120.98	117.28	180.72	287.63
		$\hat{A}_{(HS2,PPS)}$	111.10	90.27	115.62	120.89	117.75	178.65	289.68
		$\hat{A}_{(E,PPS)}$	111.12	*	*	*	*	*	*
1	0	$\hat{A}_{(U,PPS)}$	100	89.88	116.93	120.98	215.77	168.27	282.65
		$\hat{A}_{(r,PPS)}$	56.38	89.94	116.74	120.97	216.15	133.17	282.80
		$\hat{A}_{(p,PPS)}$	41.32	89.87	116.95	120.98	215.77	176.25	280.60
		$\hat{A}_{(reg,PPS)}$	102.43	89.94	116.71	120.97	219.87	113.26	283.50
		$\hat{A}_{(Bai,PPS)}$	107.62	89.87	116.95	120.98	215.78	178.69	286.36
		$\hat{A}_{(BTr,PPS)}$	89.87	89.87	116.95	120.98	215.87	174.91	288.58
		$\hat{A}_{(BTp,PPS)}$	69.65	89.91	116.83	120.97	215.78	118.44	282.80
		$\hat{A}_{(HS1,PPS)}$	118.08	89.87	116.95	120.98	215.77	180.72	287.63
		$\hat{A}_{(HS2,PPS)}$	111.10	90.27	115.62	120.89	217.27	178.65	289.68
		$\hat{A}_{(E,PPS)}$	111.12	*	*	*	*	*	*
1	1	$\hat{A}_{(U,PPS)}$	100	89.88	116.93	120.98	446.03	168.27	282.65
		$\hat{A}_{(r,PPS)}$	56.38	89.94	116.74	120.97	447.57	133.17	282.80
		$\hat{A}_{(p,PPS)}$	41.32	89.87	116.95	120.98	446.01	176.25	280.60
		$\hat{A}_{(reg,PPS)}$	102.43	89.94	116.71	120.97	463.13	113.26	283.50
		$\hat{A}_{(Bai,PPS)}$	107.62	89.87	116.95	120.98	446.05	178.69	286.36
		$\hat{A}_{(BTr,PPS)}$	89.87	89.87	116.95	120.98	446.44	174.91	288.58
		$\hat{A}_{(BTp,PPS)}$	69.65	89.91	116.83	120.97	446.07	118.44	282.80
		$\hat{A}_{(HS1,PPS)}$	118.08	89.87	116.95	120.98	446.02	180.72	287.63
		$\hat{A}_{(HS2,PPS)}$	111.10	90.27	115.62	120.89	452.19	178.65	289.68
		$\hat{A}_{(E,PPS)}$	111.12	*	*	*	*	*	*

"*"Percent relative efficiency (PRE) not considered due to unrealistic calculations.

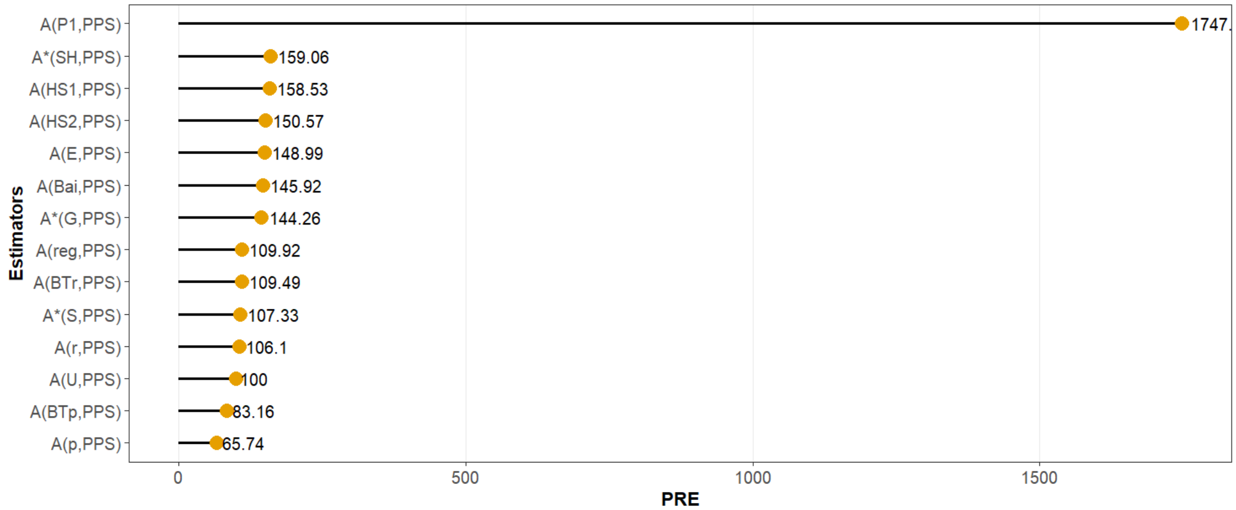

Fig. 6.1. PRE of $\hat{\hat{A}}_{(P1,PPS)}$ at a = ρ_{uv} , b = $\beta_{2(v)}$, $\alpha=0$ and $\eta=0$ for population - I

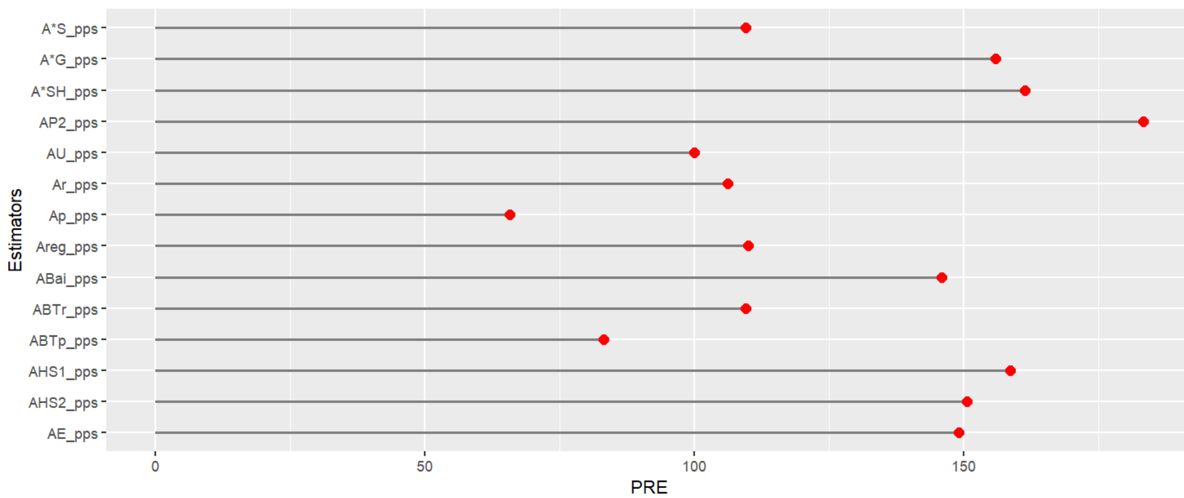

Fig. 6.2. PRE of $\hat{\hat{A}}_{(P2,PPS)}$ at a = $\beta_{2(v)}$, b = ρ_{uv} , $\alpha=0$ and $\eta=-1$ for population - I

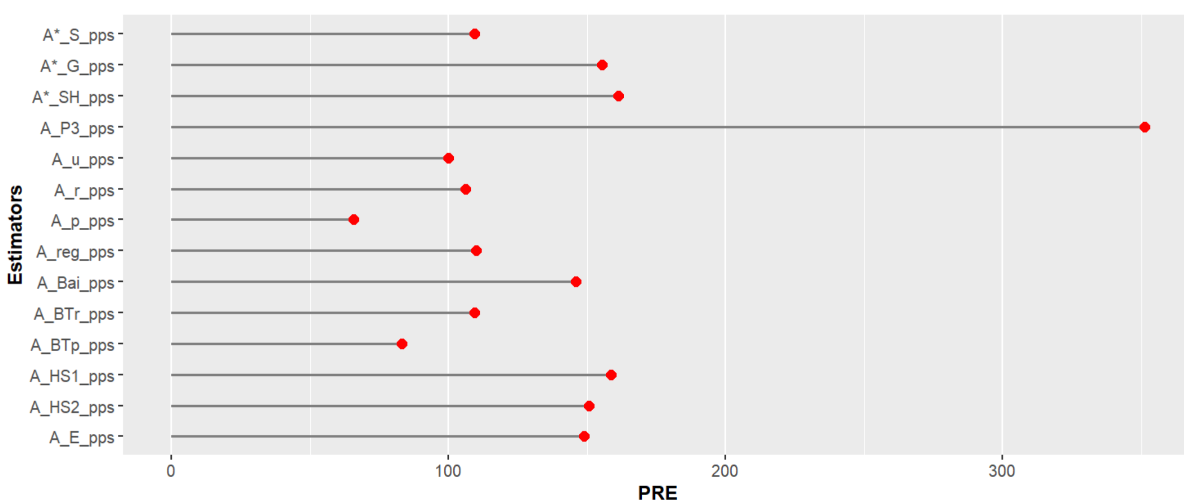

Fig. 6.3. PRE of $\hat{\hat{A}}_{(P3,PPS)}$ at a = $\beta_{(v)}$, b = C_v , $\alpha=1$ and $\eta=-1$ for population - I

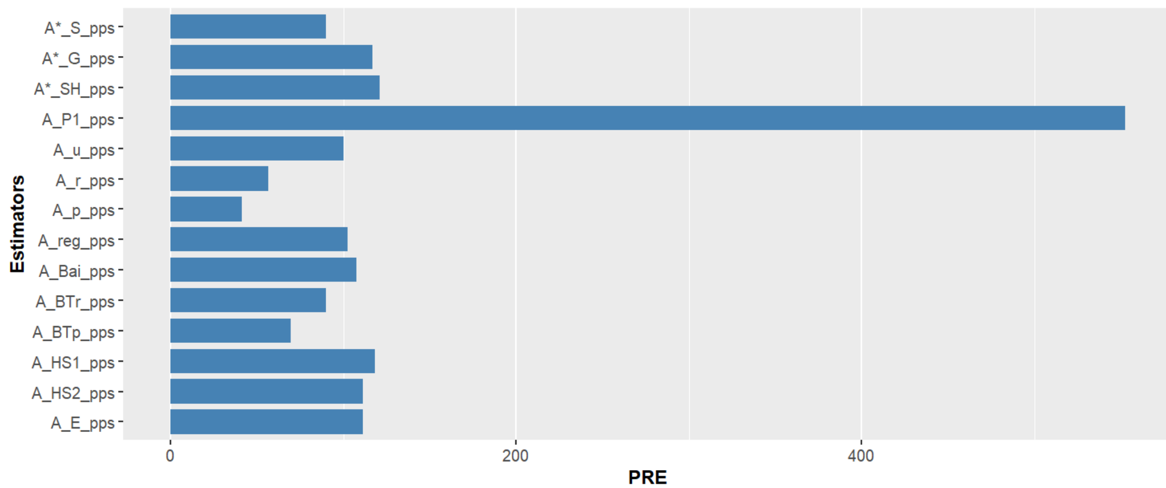

Fig. 6.4. PRE of $\hat{\hat{A}}_{(P1,PPS)}$ at a = C_v, b = $\beta_{(v)}$, $\alpha = 0.5$ and $\eta = 1$ for population - II

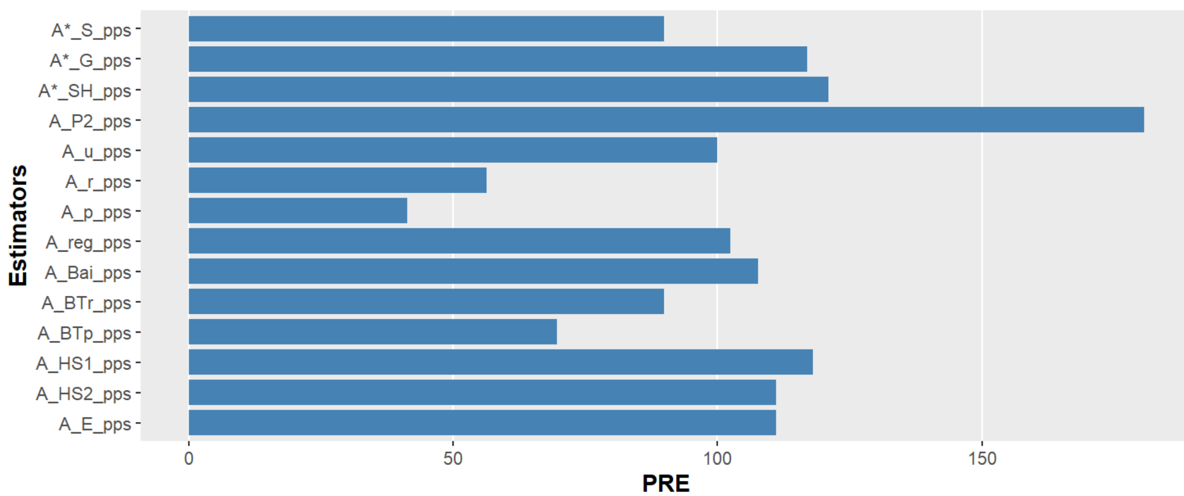

Fig. 6.5. PRE of $\hat{\hat{A}}_{(P2,PPS)}$ at a = $\beta_{2(v)}$, b = ρ_{uv}, $\alpha = 1$ and $\eta = -1$ for population - II

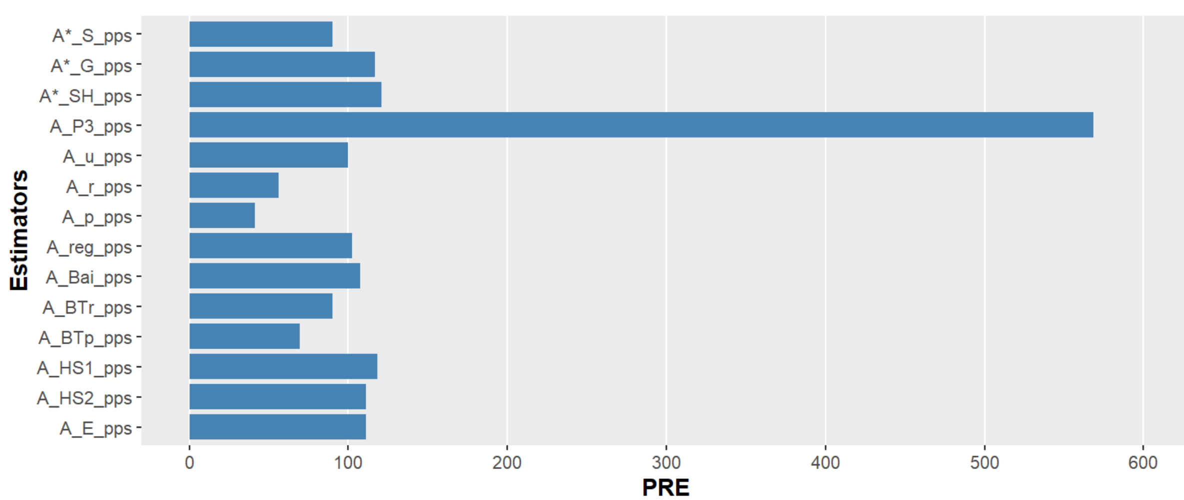

Fig. 6.6. PRE of $\hat{\hat{A}}_{(P3,PPS)}$ at a = $\beta_{2(v)}$, b = ρ_{uv}, $\alpha = 0.25$ and $\eta = 1$ for population - II

7. RESULTS

To evaluate the performance of the proposed estimators $\hat{\hat{A}}_{(Pj,PPS)}$, j = 1, 2, 3 over other existing estimators, we have used two real data-sets whose description is given in Table 6.1. By computing the percent relative efficiencies (PRE) with respect to conventional unbiased estimator $\hat{\hat{A}}_{(U,PPS)}$. The generalizability of the new developed class of estimators was also tested. The findings indicates that the proposed class of estimators is more efficient than the other estimators. We have computed the PRE's of the proposed classes of estimators $\hat{\hat{A}}_{(Pj,PPS)}$, j = 1, 2, 3 for different values of (α, η) and findings are given in Table 6.2 and 6.3. Description of the statistics of the population is represented by Table 6.1. Tables 6.2 and 6.3. It is observed from the table 6.2 and 6.3 that the proposed classes of estimators $\hat{\hat{A}}_{(Pj,PPS)}$, j = 1, 2, 3 perform well than all the other existing estimators considered here with substantial gain in efficiency. The larger gain in efficiency by using the suggested classes of estimators $\hat{\hat{A}}_{(Pj,PPS)}$, j = 1, 2, 3 over other existing estimators is looked upon at (α, η) = (0,0) for population I while for population II it is observed at (α, η) = (0.5, 1). The largest gain in efficiency (PRE =1747.38%) by using the proposed class of estimator $\hat{\hat{A}}_{(P1,PPS)}$ is seen at a = ρ_{uv}, b = $\beta_{2(v)}$, $\alpha = 0$ and $\eta = 0$ in population I while for population II the largest gain in efficiency(PRE = 552.50%) is observed at a = C_v, b = $\beta_{2(v)}$, $\alpha = 0.5$ and $\eta = 1$.

Thus, under the chosen population and selected scalars the proposed families of estimators $\hat{\hat{A}}_{(Pj,PPS)}$, j = 1, 2, 3; can achieve higher percent relative efficiencies (PREs) than the listed competitors when their constants are optimized.

The percent relative efficiency of the proposed classes is estimators is also shown by the figures 6.1, 6.2, 6.3, 6.4, 6.5 and 6.6 for the values at which the proposed estimators are most efficient.

Thus, it is recommended that the proposed study is useful to the practitioners and researchers engaged in this particular area of research.

8. CONCLUSION

In this article, we proposed three new generalized class of estimators which utilizes two auxiliary variables based on probability proportional to size scheme. From three proposed classes of estimators, thirty new estimators are identified which are demonstrated by Table 4.1, 4.2. and 4.3. We have derived the conditions under which developed families of estimators $\hat{\hat{A}}_{(Pj,PPS)}$, j = 1, 2, 3; are better than existing estimators see inequalities (5.1) – (5.13) . The proposed classes of estimators are also studied numerically and the results are represented by tables 6.2 and 6.3. On the basis of observation, it is concluded that the developed estimators out-performs the traditional and other well – known estimators. Therefore, the developed estimators are recommended for use in diverse fields of life.

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