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Some Series of Resolvable Semi–Regular Group Divisible and Rectangular Designs

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SUMMARY

Here, some series of resolvable global decomposable semi–regular group divisible designs are obtained using difference matrices. A series of resolvable rectangular designs based on difference matrix is also provided. A table of resolvable rectangular designs under $r, k \le 10$ is also provided.

Keywords: Global Decomposable design; Group divisible design; Rectangular design; Difference matrix; Resolvable design.

MSC: 62K10; 05B05; 94A62; 94B05

1. INTRODUCTION

A block design $D(v,b,r,k)$ is an arrangement of a set *X*of *v* treatments into *b* subsets (blocks) each of size *k* such that each treatment of *X* appears *r* times. The parameters satisfy the relation $bk = vr$.

A block design $D(v,b,r,k)$ is *resolvable* if the *b* blocks, each of size *k*, can be grouped into *r* resolution (or parallel) classes such that

(i) Each resolution class contains *b*/*r* blocks;

(ii) Every treatment is replicated exactly once in each resolution class.

Alternatively, if the incidence matrix *N* of $D(v,b,r,k)$ may be partitioned into sub matrices as: $N = (N_1|N_2|\cdots|N_r)$ where each $N_i\,(1 \le i \le r)$ is a $v \times (v/k)$ matrix such that each row sum of N_i is 1, then the design is resolvable.

Example 1: Consider a resolvable design $D(8,12,6,4)$ whose blocks are written column–wise and resolution classes are separated by a vertical line as follows:

$$\left(\begin{array}{cc|cc|cc|cc|cc|cc} 1 & 2 & 1 & 2 & 1 & 3 & 1 & 3 & 1 & 2 & 2 & 1 \\ 3 & 4 & 3 & 4 & 2 & 4 & 2 & 4 & 4 & 3 & 3 & 4 \\ 5 & 6 & 5 & 6 & 5 & 7 & 5 & 7 & 5 & 6 & 6 & 5 \\ 7 & 8 & 7 & 8 & 6 & 8 & 6 & 8 & 8 & 7 & 7 & 8 \end{array}\right).$$

Then its incidence matrix *N* can be partitioned as:

$$N = (N_1|N_2|N_3|N_4|N_5|N_6) = \left(\begin{array}{cc|cc|cc|cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right).$$

Clearly each $N_i\,(1 \le i \le 6)$ is a 8×2 matrix and also each row sum of N_i is one.

A balanced incomplete block (BIB) design is an arrangement of v treatments into b blocks such that

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(i) Every block contains $k(<v)$ distinct treatments;
(ii) Every treatment is replicated r times and any pair of distinct treatments occurs together in exactly λ blocks.

Further, a BIB design is symmetric if $v=b$. The positive integers: v,b,r,k,λ are known as parameters of the BIB design and they are connected by the relations:

$$bk=vr, r(k-1)=\lambda(v-1), b=\frac{\lambda v(v-1)}{k(k-1)}.$$

Following Ionin and Shrikhande (2006): the method of global decomposition produces an incidence matrix of symmetric balanced incomplete block (BIB) design as a block matrix with each block being a zero matrix or an incidence matrix of the smaller symmetric BIB design. They used generalized weighing matrices over cyclic groups and generalized Hadamard matrices over elementary abelian groups as main tools for the constructions of symmetric BIB designs.

Here, the idea of global decomposition is extended to group divisible designs and a series of such designs is obtained.Some recent constructions of global decomposable partially balanced designs are described in Saurabh(2026*a*). A method of construction of resolvable rectangular designs is also described using difference matrices.

Notations: SRX numbers used here are from Clatworthy (1973). I_n is the identity matrix of order n , $J_{v\times b}$ is an $v\times b$ matrix with all entries 1, in particular $J_{v\times v}=J_v.N^T$ is the transpose of the matrix $N, C_p=\{1,\alpha,\alpha^2,\cdots,\alpha^{p-1}\}$ is a cyclic group of order *p*, where 1 is the identity element and

$EA(q=p^n)\approx \underbrace{C_p\times C_p\times\cdots\times C_p}_{n \text{ times}}$ denotes the elementary abelian group of order *q* and $\beta=circ(0100\ldots0)$ is a permutation circulant matrix of order n such that $\beta^n=I_n$. Clearly β is an orthogonal matrix and hence each row and column sum of β is one and $\beta\beta^T=I_n$.

A square block matrix is a block–circulant matrix in which each block is a circulant matrix of smaller order. Further, if $A=(a_{ij})$ is an $m\times n$ matrix and $B=(b_{ij})$ is a $p\times q$ matrix, then their Kronecker product $A\otimes B$ is the $mp\times nq$ block matrix:

$$A\otimes B=\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}.$$

Some other relevant definitions in the context of the paper are as follows:

For each pair of distinct rows: $(x_1,x_2,\ldots,x_b)$ and $(y_1,y_2,\ldots,y_b)$ of a matrix over a multiplicative group, the multi–set $\{x_iy_i^{-1}: i=1,2,\ldots,b\}$ denotes their inner product. Let $\lambda(\geq 1)$ and $m,n(\geq 2)$ be positive integers and $\mathcal{G}$ be a multiplicatively written group consisting of m elements. Then a *difference matrix* $D(\lambda m,n;m)$ is a $\lambda m\times n$ matrix with entries from $\mathcal{G}$ such that the inner product of any two distinct rows of $D(\lambda m,n;m)$ contains each element of $\mathcal{G}$ exactly λ times.

A square difference matrix of order $n(=\lambda m)$ is also known as a *Generalized Hadamard matrix*, $GH(n;m)$ over a group $\mathcal{G}$ containing m elements. The groups used in this paper are cyclic group C_p and elementary abelian group.

Consider an arrangement of $v=mn$ treatments in an $m\times n$ array. A *semi–regular group divisible design* is an arrangement of these treatments in b blocks, each of size k such that:

1. Every treatment appears no more than once in a block;
2. Every treatment is distributed across exactly r blocks;
3. Every pair of treatments in the same row of the $m\times n$ array occurs together in λ_1 blocks, while the remaining pairs of treatments occur together in λ_2 blocks;

4. $r-\lambda_1>0, rk-v\lambda_2=0.$

Further, let N be $v\times b$ incidence matrix of a block design such that $J_v N = kJ_{v\times b}$, which ensures that the each column sum of N is k, and satisfies the following conditions (i) or (ii):

(i) $NN^T=(r-\lambda_1)(I_m\otimes I_n)+(\lambda_1-\lambda_2)(I_m\otimes J_n)+\lambda_2(J_m\otimes J_n).$

(ii) $NN^T=(r-\lambda_2)(I_n\otimes I_m)+\lambda_2(J_n\otimes J_m)+(\lambda_1-\lambda_2)\{(J_n-I_n)\otimes I_m\}.$

The above condition (i) or (ii) ensures that each row sum of N is r and the concurrence numbers are λ_1 and λ_2. Hence, N represents a group divisible design with parameters: $v=mn, r, k, b, \lambda_1, \lambda_2, m, n$ and they satisfy the relations: $bk=vr; (n-1)\lambda_1+n(m-1)\lambda_2=r(k-1)$. For corresponding group divisible association schemes or arrangement of $v=mn$ treatments in the $m\times n$ array, we refer to Saurabh (2024).

2. GLOBAL DECOMPOSABLE GROUP DIVISIBLE DESIGNS

Lemma 1 [Gibbons and Mathons (1987)**]:** The existence of a difference matrix $D(\lambda m, n; m)$ over a multiplicative group $\mathcal{G}$ of order m implies the existence of a resolvable group divisible design with parameters:

$$v=\lambda m^2, b=mn, r=n, k=\lambda m, \lambda_1=0, \lambda_2=\lambda, m^*=\lambda m, n^*=m. \quad (1)$$

Proof: Consider a difference matrix $M=D(\lambda m, n; m)$ of order $\lambda m\times n$ over a multiplicative group $\mathcal{G}$. Let N be a $\lambda m^2\times mn$ matrix obtained from replacing the group elements of M by corresponding $m\times m$ permutation matrices. Then using properties of permutation matrices, we have

(i) $J_{\lambda m^2}N=(\lambda m)J_{\lambda m^2\times mn}$;

(ii) $NN^T=n(I_{\lambda m}\otimes I_m)-\lambda(I_{\lambda m}\otimes J_m)+\lambda(J_{\lambda m}\otimes J_m).$

Hence, N represents a group divisible design with above mentioned parameters (1). Moreover, as each row of the permutation matrix sums to unity, the above series of designs is resolvable.

Theorem 1: Let p be a prime. Then there exists a resolvable global decomposable semi–regular group divisible design with parameters:

$$v=b=p^3, r=k=p^2, \lambda_1=0, \lambda_2=p, m=p^2, n=p. \quad (2)$$

Proof. Let $C_p=\{1,\alpha,\alpha^2,\cdots,\alpha^{p-1}\}$ be a cyclic group of order p and $\beta=circ(0100\ldots0)$ be a permutation circulant matrix of order p such that $\beta^p=I_p$. Then, we obtain semi–regular group divisible designs with parameters (2) via following steps:

Step 1: Consider the following block circulant matrix:

$$N=\begin{pmatrix} I_p & I_p & I_p & \cdots & I_p \\ I_p & \beta & \beta^2 & \cdots & \beta^{p-1} \\ I_p & \beta^2 & (\beta^2)^2 & \cdots & (\beta^{p-1})^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_p & \beta^{p-1} & (\beta^2)^{p-1} & \cdots & (\beta^{p-1})^{p-1} \end{pmatrix}$$

Then, we have

$$J_{p^2}N=pJ_{p^2}; NN^T=p(I_p\otimes I_p)-(I_p\otimes J_p)+(J_p\otimes J_p).$$

Hence, we obtain a semi–regular group divisible design with parameters:

$$v=b=p^2, r=k=p, \lambda_1=0, \lambda_2=1, m=n=p. \quad (3)$$

Moreover, as each row of the permutation circulant matrix sums to unity, the designs with parameters (3) are resolvable.

Step 2: Replace $I_p\to 1, \beta^i\to\alpha^i\ (1\le i\le p-1)$ in N to obtain a difference matrix $D(p,p;p)$. Then, the Kronecker product $D(p,p;p)\otimes D(p,p;p)$ represents another difference matrix $D(p^2,p^2;p)$ whose each internal smaller block represents a $D(p,p;p)$.

Step 3:Replace $1 \to I_p, \alpha^i \to \beta^i \left(1 \le i \le p-1\right)$ in $D\left(p^2, p; p\right)$ to obtain a $p^3 \times p^3$ symmetric matrix, say M. Then, we have

$$J_{p^3} M = p^2 J_{p^3}; MM^T = p^2\left(I_{p^2} \otimes I_p\right) - p\left(I_{p^2} \otimes J_p\right) + p\left(J_{p^2} \otimes J_p\right).$$

Hence, we obtain a global decomposable semi–regular group divisible design with parameters(2) whose each internal smaller block matrix of size $p^2 \times p^2$ represents a resolvable semi–regular group divisible design with parameters (3).Moreover, as each row of the permutation circulant matrix sums to unity, the designs with parameters (2) are also resolvable.

Example 2: Let $N = \begin{pmatrix} I_3 & I_3 & I_3 \\ I_3 & \beta & \beta^2 \\ I_3 & \beta^2 & \beta \end{pmatrix}$, where $\beta = circ(010)$ is a permutation circulant matrix of order 3. Then, we have

$$J_9 N = 3J_9; NN^T = 3\left(I_3 \otimes I_3\right) - \left(I_3 \otimes J_3\right) + \left(J_3 \otimes J_3\right).$$

Hence, we obtain a resolvable semi–regular group divisible design *SR*23 with parameters: $v = b = 9, r = k = 3, \lambda_1 = 0, \lambda_2 = 1, m = n = 3.$

Further, replacing $I_3 \to 1, \beta \to \alpha, \beta^2 \to \alpha^2$ in N, we obtain

$$D(3,3;3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{pmatrix}$$ over a cyclic group $C_3 = \left\{1, \alpha, \alpha^2\right\}$.

Then the Kronecker product $D(3,3;3) \otimes D(3,3;3)$ represents $D(9,9;3)$ as given below:

$$D(9,9;3) = \left(\begin{array}{ccc|ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & 1 & \alpha & \alpha^2 & 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha & 1 & \alpha^2 & \alpha & 1 & \alpha^2 & \alpha \\ \hline 1 & 1 & 1 & \alpha & \alpha & \alpha & \alpha^2 & \alpha^2 & \alpha^2 \\ 1 & \alpha & \alpha^2 & \alpha & \alpha^2 & 1 & \alpha^2 & 1 & \alpha \\ 1 & \alpha^2 & \alpha & \alpha & 1 & \alpha^2 & \alpha^2 & \alpha & 1 \\ \hline 1 & 1 & 1 & \alpha^2 & \alpha^2 & \alpha^2 & \alpha & \alpha & \alpha \\ 1 & \alpha & \alpha^2 & \alpha^2 & 1 & \alpha & \alpha & \alpha^2 & 1 \\ 1 & \alpha^2 & \alpha & \alpha^2 & \alpha & 1 & \alpha & 1 & \alpha^2 \end{array}\right),$$

whose each internal smaller block represents $D(3,3;3)$. Further, replacing $1 \to I_3, \alpha \to \beta, \alpha^2 \to \beta^2$ in $D(9,9;3)$, we obtain a 27×27 matrix:

$$M = \left(\begin{array}{ccc|ccc|ccc} I_3 & I_3 & I_3 & I_3 & I_3 & I_3 & I_3 & I_3 & I_3 \\ I_3 & \beta & \beta^2 & I_3 & \beta & \beta^2 & I_3 & \beta & \beta^2 \\ I_3 & \beta^2 & \beta & I_3 & \beta^2 & \beta & I_3 & \beta^2 & \beta \\ \hline I_3 & I_3 & I_3 & \beta & \beta & \beta & \beta^2 & \beta^2 & \beta^2 \\ I_3 & \beta & \beta^2 & \beta & \beta^2 & I_3 & \beta^2 & I_3 & \beta \\ I_3 & \beta^2 & \beta & \beta & I_3 & \beta^2 & \beta^2 & \beta & I_3 \\ \hline I_3 & I_3 & I_3 & \beta^2 & \beta^2 & \beta^2 & \beta & \beta & \beta \\ I_3 & \beta & \beta^2 & \beta^2 & I_3 & \beta & \beta & \beta^2 & I_3 \\ I_3 & \beta^2 & \beta & \beta^2 & \beta & I_3 & \beta & I_3 & \beta^2 \end{array}\right).$$

Then, we have

$$J_{27} M = 9J_{27};$$
$$MM^T = 9\left(I_9 \otimes I_3\right) - 3\left(I_9 \otimes J_3\right) + 3\left(J_9 \otimes J_3\right).$$

Hence, M represents a resolvable global decomposable semi–regular group divisible design *SR*102: $v = b = 27, r = k = 9, \lambda_1 = 0, \lambda_2 = 3, m = 9, n = 3$ whose each internal block matrix represents another resolvable semi–regular group divisible design *SR*23: $v = b = 9, r = k = 3, \lambda_1 = 0, \lambda_2 = 1, m = n = 3.$

Example 3: Let $N=\begin{pmatrix} I_5 & I_5 & I_5 & I_5 & I_5 \\ I_5 & \beta & \beta^2 & \beta^3 & \beta^4 \\ I_5 & \beta^2 & \beta^4 & \beta & \beta^3 \\ I_5 & \beta^3 & \beta & \beta^4 & \beta^2 \\ I_5 & \beta^4 & \beta^3 & \beta^2 & \beta \end{pmatrix}$,

where $\beta = circ(01000)$ is a permutation circulant matrix of order 5. Then, we have

$$J_{25}N = 5J_{25}; NN^T = 5(I_5 \otimes I_5) - (I_5 \otimes J_5) + (J_5 \otimes J_5).$$

Hence, we obtain a resolvable semi–regular group divisible design *SR*60 with parameters: $v = b = 25, r = k = 5, \lambda_1 = 0, \lambda_2 = 1, m = n = 5.$

Moreover, replacing $I_5 \to 1, \beta \to \alpha, \beta^2 \to \alpha^2, \beta^3 \to \alpha^3, \beta^4 \to \alpha^4$ in *N*, we obtain

$D(5,5;5) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \alpha^2 & \alpha^4 & \alpha & \alpha^3 \\ 1 & \alpha^3 & \alpha & \alpha^4 & \alpha^2 \\ 1 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha \end{pmatrix}$ over a cyclic group $C_5 = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4\}$.

Then the Kronecker product $D(5,5;5) \otimes D(5,5;5)$ represents $D(25,25;5)$ as given below:

$D(25,25;5) = \begin{pmatrix} A_0 & A_0 & A_0 & A_0 & A_0 \\ A_0 & A_1 & A_2 & A_3 & A_4 \\ A_0 & A_2 & A_4 & A_1 & A_3 \\ A_0 & A_3 & A_1 & A_4 & A_2 \\ A_0 & A_4 & A_3 & A_2 & A_1 \end{pmatrix}$,

where each $A_i (0 \le i \le 4)$ represents a $D(5,5;5)$, where

$A_i = \alpha^i \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \alpha^2 & \alpha^4 & \alpha & \alpha^3 \\ 1 & \alpha^3 & \alpha & \alpha^4 & \alpha^2 \\ 1 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha \end{pmatrix}$.

Further, replacing $1 \to I_5, \alpha \to \beta, \alpha^2 \to \beta^2, \alpha^3 \to \beta^3, \alpha^4 \to \beta^4$ in $D(25,25;5)$, we obtain a 125×125 block matrix which satisfies the following:

$$J_{125}M = 25J_{125};$$
$$MM^T = 25(I_{25} \otimes I_5) - 5(I_{25} \otimes J_5) + 5(J_{25} \otimes J_5).$$

Hence, *M* represents a resolvable global decomposable semi–regular group divisible design with parameters: $v = b = 125, r = k = 25, \lambda_1 = 0, \lambda_2 = 5, m = 25, n = 5,$ whose each internal block matrix represents another resolvable semi–regular group divisible design *SR*60: $v = b = 25, r = k = 5, \lambda_1 = 0, \lambda_2 = 1, m = n = 5.$

Theorem 2: Let *q* be an odd prime or odd prime power. There exists a resolvable global decomposable semi–regular group divisible design with parameters:

$$v = b = 2q^2, r = k = 2q, \lambda_1 = 0, \lambda_2 = 2, m = 2q, n = q. \quad (4)$$

Proof: It is known that there exists a difference matrix, $D(2q,2q;q) = [D_{ik}]_{i,k=1,2} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ where each D_{ik} is a difference matrix, $D(q,q;q)$ over an elementary abelian group $EA(q)$ [vide Lemma 1].

Let $N_{ik}(i,k=1,2)$ be $q^2 \times q^2$ (0,1) – matrices obtained by replacing the group elements in D_{ik} by the corresponding $q \times q$ permutation matrices.

Then, $N = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$ represents a semi–regular group divisible design with parameters (3) where each N_{ik} represents a semi–regular group divisible design with parameters [vide Lemma 1]:

$$v = b = q^2, r = k = q, \lambda_1 = 0, \lambda_2 = 1, m = n = q. \quad (5)$$

Moreover, as each row of $q \times q$ permutation matrices sums to unity, the above series (5) of designs is resolvable. Hence, we obtain a resolvable global decomposable semi–regular group divisible design with parameters (4) whose internal smaller block

matrices of size $q^2 \times q^2$ represents a resolvable semi–regular group divisible design with parameters (5).

Example 4: Let $C_5 = \{1, \alpha, \alpha^2, \alpha^3, \alpha^4\}$ be a cyclic group. Then $D(10,10;5) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ is a difference matrix over C_5 where each $D_{ik}; i,k = 1,2$ is a $D(5,5;5)$ as given below [see Dawson (1985) and Dey and Mukerjee (1999)]:

$$D_{11} = \begin{pmatrix} 1 & \alpha & \alpha^4 & \alpha^4 & \alpha \\ 1 & \alpha^4 & 1 & \alpha^3 & \alpha^3 \\ 1 & \alpha^2 & \alpha & \alpha^2 & 1 \\ 1 & 1 & \alpha^2 & \alpha & \alpha^2 \\ 1 & \alpha^3 & \alpha^3 & 1 & \alpha^4 \end{pmatrix},$$

$$D_{21} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \alpha^3 & \alpha & \alpha^4 & \alpha^2 \\ 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \\ 1 & \alpha^4 & \alpha^3 & \alpha^2 & \alpha \\ 1 & \alpha^2 & \alpha^4 & \alpha & \alpha^3 \end{pmatrix},$$

$$D_{12} = \begin{pmatrix} 1 & \alpha^2 & \alpha^3 & \alpha^3 & \alpha^2 \\ \alpha & \alpha^4 & \alpha & \alpha^2 & \alpha^2 \\ \alpha^4 & \alpha^3 & \alpha & \alpha^3 & \alpha^4 \\ \alpha^4 & \alpha^4 & \alpha^3 & \alpha & \alpha^3 \\ \alpha & \alpha^2 & \alpha^2 & \alpha & \alpha^4 \end{pmatrix},$$

$$D_{22} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \alpha^3 & \alpha & \alpha^4 & \alpha^2 & 1 \\ \alpha^2 & \alpha^3 & \alpha^4 & 1 & \alpha \\ \alpha^2 & \alpha & 1 & \alpha^4 & \alpha^3 \\ \alpha^3 & 1 & \alpha^2 & \alpha^4 & \alpha \end{pmatrix}.$$

Moreover, replacing the group elements of C_5 by the corresponding permutation matrices: $1 \to I_5, \alpha \to circ(01000)$, we obtain a resolvable semi–regular group divisible design with parameters: $v = b = 50, r = k = 10, \lambda_1 = 0, \lambda_2 = 2, m = 10, n = 5$ [see Saurabh and Sinha (2023*a*)] where each D_{ik} corresponds to a resolvable semi–regular group divisible design SR60 with parameters: $v = b = 25, r = k = 5, \lambda_1 = 0, \lambda_2 = 1, m = n = 5.$ Hence, we obtain a resolvable global decomposable group divisible design.

3. RESOLVABLE RECTANGULAR DESIGNS

Consider an arrangement of $v = mn$ treatments in an $m \times n$ array 'A' (see given below). A rectangular design is an arrangement of these treatments into b blocks, each of size k such that:

1. Every treatment appears no more than once in a block;
2. Every treatment is distributed across exactly r blocks;
3. Every pair of distinct treatments in the same row of array A occurs together in λ_1 blocks; those in the same column occur together in λ_2 blocks; and remaining pairs of distinct treatments occur together in λ_3 blocks.

The treatments in the same row of the array A are first associates, the treatments in the same column are second associates and the treatments not belonging to the same row or column are third associates.

The non–negative integers: $v = mn, b, r, k, \lambda_1$, λ_2 and λ_3 are known as parameters of the rectangular design and they satisfy the relations: $bk = vr; (n-1)\lambda_1 + (m-1)\lambda_2 + (n-1)(m-1)\lambda_3 = r(k-1).$ A rectangular design is an important family of 3–associate–class partially balanced incomplete block (PBIB) design.

Further, let N be a $v \times b$ incidence matrix of a block design which satisfies the following conditions:

(i) $J_v N = kJ_{v \times b}$, which ensures that the each column sum of N is k;

(ii) $NN^T = (r - \lambda_1)(I_m \otimes I_n) + \lambda_1 (I_m \otimes J_n) + \lambda_2 \{(J_m - I_m) \otimes I_n\} + \lambda_3 \{(J_m - I_m) \otimes (J_n - I_n)\},$

which ensures that each row sum of N is r and the concurrence numbers are λ_1, λ_2 and λ_3.

Hence, N represents a rectangular design with parameters: $v=mn, r, k, b, \lambda_1, \lambda_2, \lambda_3, m, n$ [see Singh and Saurabh (2023)].

The $m \times n$ array A is given as:

$$\begin{matrix} 0 & 1 & 2 & \cdots & n-1 \\ n & n+1 & n+2 & \cdots & 2n-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (m-1)n & (m-1)n+1 & (m-1)n+2 & \cdots & mn-1 \end{matrix}$$

Remark: A rectangular design reduces to a group divisible design if $\lambda_1 = \lambda_3$ (by combining first and third associates) or $\lambda_2 = \lambda_3$ (by combining second and third associates). The arrangement of $v = mn$ elements in $m \times n$ array of these group divisible designs may be found in Saurabh and Sinha (2023*b*) and Saurabh (2024).

Let $\Gamma_i \left(1 \le i \le n\right)$ be a $n \times n$ matrix whose i^{th} column contains only 1's and 0 elsewhere. Then

(i) $\sum_{i=1}^{n} \Gamma_i = J_n$ (ii) $\Gamma_i \Gamma_j^T = \begin{cases} J_n, i = j \\ O_n, i \ne j \end{cases}$, where O_n is a null matrix of order n.

Example 5: For $n = 3$, $\Gamma_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,

$\Gamma_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \Gamma_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Given below is a series of resolvable rectangular designs:

Theorem 3: Let p be a prime. Then there exists a resolvable rectangular design with parameters:

$$v = p^2, b = p\left\{\mu + s + t\left(p-1\right)\right\}, r = \mu + s + t\left(p-1\right),$$

$$k = p, \lambda_1 = s, \lambda_2 = \mu, \lambda_3 = t, m = n = p. \qquad (6)$$

Proof. Consider the following block matrices:

(i) $N_1 = \mu$ copies of block matrix $\begin{pmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{pmatrix}$ arranged column–wise;

(ii) $N_2 = s$ copies of block matrix $\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_p \end{pmatrix}$ arranged column–wise, where $\Gamma_i \left(1 \le i \le p\right)$ is a $p \times p$ matrix whose i^{th} column contains only 1's and 0 elsewhere;

(iii) Let $\beta = circ\left(010\ldots0\right)$ be permutation circulant matrix of order p.

Consider the following difference matrix:

$$D\left(p, p; p\right) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \alpha & \alpha^2 & \ldots & \alpha^{p-1} \\ 1 & \alpha^2 & \left(\alpha^2\right)^2 & \ldots & \left(\alpha^{p-1}\right)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha^{p-1} & \left(\alpha^2\right)^{p-1} & \ldots & (\alpha^{p-1})^{p-1} \end{pmatrix}$$

over $C_p = \left\{1, \alpha, \alpha^2, \cdots, \alpha^{p-1}\right\}$.

Further, delete first column of $D\left(p, p; p\right)$ and replace group elements by the corresponding permutation matrices in the column deleted matrix to obtain a block matrix M as given below:

$$M = \begin{pmatrix} I_p & I_p & \ldots & I_p \\ \beta & \beta^2 & \ldots & \beta^{p-1} \\ \beta^2 & \left(\beta^2\right)^2 & \ldots & \left(\beta^{p-1}\right)^2 \\ \vdots & \vdots & \vdots & \vdots \\ \beta^{p-1} & \left(\beta^2\right)^{p-1} & \ldots & (\beta^{p-1})^{p-1} \end{pmatrix}.$$

Let $N_3 = t$ copies of block matrix M arranged column–wise.

Consider the following block matrix $N=\left(N_1|N_2|N_3\right)$ obtained by juxtaposing the matrices N_1, N_2 and N_3. Then, we have

(i) $J_{p^2}N = pJ_{p^2\times\{p(\mu+s+t(p-1)\}}$;

(ii) $NN^T = N_1N_1^T + N_2N_2^T + N_3N_3^T$

$$= \mu\left(J_p \otimes I_p\right) + s\left(I_p \otimes J_p\right) + t\{\left(p-1\right)\left(I_p \otimes I_p\right) + \left(J_p - I_p\right)\otimes\left(J_p - I_p\right)$$

$$= \left(\mu + t\left(p-1\right)\right)\left(I_p \otimes I_p\right) + s\left(I_p \otimes J_p\right) + \mu\left\{\left(J_p - I_p\right)\otimes I_p\right\} + t\left\{\left(J_p - I_p\right)\otimes\left(J_p - I_p\right)\right\}.$$

Hence, N representsthe incidence matrix of a rectangular design with the parameters (6) mentioned above.Moreover, since each row sum of the block matrices $\begin{pmatrix} I_p \\ I_p \\ \vdots \\ I_p \end{pmatrix}, \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_p \end{pmatrix}$ and $p\times p$ permutation matrices is one, the design is resolvable.

Example 6: For $p=5, s=1, \mu=2, t=3,$ we obtain a resolvable rectangular design with parameters: $v=25, b=75, r=15, k=5, \lambda_1=1, \lambda_2=2, \lambda_3=3, m=n=5.$

A Table 1 of resolvable rectangular designs under $r,k\le 10$, obtained using Theorem 3 for different choices of s, μ and t, is presented below:

Table 1. Resolvable rectangular designs under $r,k\le 10$

No.	v	m	n	b	r	k	λ_1	λ_2	λ_3
1	9	3	3	12	4	3	0	2	1
2	15	3	5	45	9	3	0	1	2
3	25	5	5	30	6	5	0	2	1
4	25	5	5	35	7	5	0	3	1
5	25	5	5	40	8	5	0	4	1
6	25	5	5	45	9	5	0	5	1
7	25	5	5	45	9	5	0	1	2
8	25	5	5	50	10	5	0	6	1
9	25	5	5	45	9	5	2	3	1
10	25	5	5	50	10	5	2	4	1
11	49	7	7	56	8	7	0	2	1
12	49	7	7	63	9	7	0	3	1
13	49	7	7	70	10	7	0	4	1

4. CONCLUSION

Here, some series of global decomposable semi–regular group divisible designs are described using difference matrices. A series of resolvable rectangular designs is also obtained using difference matrices and permutation circulant matrices. Further, some resolvable rectangular designs under $r,k\le 10$ are also tabulated. A recent survey on rectangular designs is available in Singh and Saurabh (2023). Some series of cyclic rectangular designs are obtained by Sinha *et al.* (2002) and Saurabh (2026*b*), among others.

Global decomposable designs may have possible applications in cryptography and coding theory. Group divisible designs already have well–known applications in group testing experiments and intercropping experiments [see Raghavarao and Padgett (2005)]. Group divisible designs are also applicable in agricultural field trials.

Resolvable group divisible designs are used in the construction of A^2–codes and low–density parity–check (LDPC) codes [see Pei (2006) and Xu *et al.* (2015, 2020)]. The resolvability can likewise be useful in multi–site experiments and in experiments with multiple individuals handling experimental runs. Such situations exist in industrial, agricultural and forestry experiments. For details, we refer to Morgan and Reck (2007), John and Williams (1995), and Patterson and Silvey (1980).

Rectangular designs are useful as factorial experiments, which have balance as well as orthogonality [see Gupta and Mukerjee (1989)]. In addition, rectangular designs having $\lambda_3 > \lambda_1, \lambda_2$ when used as the $m\times n$ complete confounded experiments, the loss of information on the main effects is small [see Suen (1989)].Recently rectangular designs have been used in the constructions of LDPC codes and threshold schemes [see Saurabh (2026*c*)].

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DECLARATIONS

- **Conflict of Interest**

  The author declares that he has no conflict of interest.

- **Data Availability**

  Not Applicable.

- **Ethical Approval**

  Ethical approval was not required for this study because it did not involve human participants, animals, or collection of identifiable personal information.

- **Clinical Trial Number**

  Not Applicable. This study did not involve a clinical trial.

- **Consent To Publish**

  Not applicable. This study did not involve the recruitment or participation of human subjects.

- **Consent To Participate**

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