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Generalized Classes of Neutrosophic Auxiliary Information Estimators in Survey Sampling: A Review of Unified Framework

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SUMMARY

The use of auxiliary information plays an important role in enhancing the accuracy of estimating the population mean in survey sampling. However, in many practical situations, the available information may be imprecise, incomplete, or indeterminate, making the classical statistical framework inadequate for representing such uncertainty. Neutrosophic statistics provides a suitable framework for handling data involving indeterminacy and ambiguity. Motivated by the general class suggested by Srivastava (1971) and the broader class introduced by Singh and Upadhyaya (1986), this study presents a systematic review and unified organization of general and almost-unbiased classes of estimators for finite population mean estimation under a neutrosophic setting. The resulting unified framework provides a general structure within which several ratio, product, exponential, logarithmic, and regression-type neutrosophic estimators can be represented as particular cases. The unified formulation establishes systematic relationships among existing estimator forms and extends the classical theory of auxiliary-information-based estimation to settings involving indeterminate and ambiguous data. Theoretical expressions for the bias and mean squared error (MSE) of the proposed general classes are derived, and their optimal forms are obtained under the neutrosophic framework. The unified formulation provides a flexible theoretical basis for constructing and analyzing estimators when auxiliary information is subject to indeterminacy and ambiguity.

Keywords: Neutrosophic statistics; General class of estimators; MSE; Auxiliary information; Bias.

1. INTRODUCTION

Classical statistics deals with data represented by precise or crisp numerical values, where no uncertainty is assumed in the measurement of observations. However, in many real-life situations, indeterminacy, vagueness, ambiguity, or imprecision may exist in the available data. For example, crop yield may be uncertain due to incomplete harvesting or measurement errors. Soil nutrient levels, such as nitrogen, phosphorus, and potassium, may vary across a field and may be expressed as interval-valued measurements due to such variation. Similarly, the severity of pest or disease infestation may be difficult to determine precisely because field observations can be incomplete, imprecise, or ambiguous. In such cases, conventional statistical techniques may not be sufficient to properly handle these types of uncertain observations. Therefore, new statistical approaches are required to deal with indeterminate and ambiguous data. Fuzzy logic is one such approach that can handle ambiguous, imprecise, or uncertain observations; however, it does not explicitly account for indeterminacy. In this context, neutrosophic techniques provide a more suitable framework, as they can address randomness as well as indeterminacy present in the data. Neutrosophic statistics serves as a flexible extension of classical statistics for analyzing data characterized by uncertainty and indeterminacy.

Neutrosophic quantitative observations can be represented by interval-valued neutrosophic numbers (Smarandache, 2014). Following Yadav and Smarandache (2023), a neutrosophic variable is represented in interval form as $Z_N = Z_L + Z_U I_N$, where $'Z_L'$ and $'Z_U'$ denote its lower and upper values,

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respectively. Consider an SRSWOR sample containing n_N units drawn from a finite population comprising $N_N = (N_L, N_U)$ units, with $\bar{y}_N \in [\bar{y}_L, \bar{y}_U]$ and $\bar{x}_N \in [\bar{x}_L, \bar{x}_U]$ denoting the sample means of the neutrosophic study variable Y_N and auxiliary variable X_N, respectively. Their corresponding population means are denoted by $\bar{Y}_N \in [\bar{Y}_L, \bar{Y}_U]$ and $\bar{X}_N \in [\bar{X}_L, \bar{X}_U]$, and their population standard deviations are denoted by $S_{yN} \in [S_{yL}, S_{yU}]$ and $S_{xN} \in [S_{xL}, S_{xU}]$, respectively. The neutrosophic coefficients of variation (CV) of Y_N and X_N are denoted $C_{yN} \in [C_{yL}, C_{yU}]$ and $C_{xN} \in [C_{xL}, C_{xU}]$, respectively, while $\rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}]$ represents their neutrosophic correlation coefficient. The neutrosophic coefficient of kurtosis of the auxiliary variable is denoted by $\beta_{2(x)N} \in [\beta_{2(x)L}, \beta_{2(x)U}]$. The corresponding neutrosophic error terms are defined as ϵ_{0N} and ϵ_{1N}.

To derive the bias and MSE of the estimator, we express

$$\bar{y}_N = \bar{Y}_N(1+\epsilon_{0N}),\ \bar{x}_N = \bar{X}_N(1+\epsilon_{1N}),$$

$$E(\epsilon_{0N}) = E(\epsilon_{1N}) = 0 \text{ and}$$

$$E(\epsilon_{0N}^2) = \theta_N C_{yN}^2, \qquad E(\epsilon_{1N}^2) = \theta_N C_{xN}^2,$$

$$E(\epsilon_{0N}\epsilon_{1N}) = \theta_N \rho_{yxN} C_{yN} C_{xN}.$$

Where, $\epsilon_{0N} \in [\epsilon_{0L}, \epsilon_{0U}]$, $\epsilon_{1N} \in [\epsilon_{1L}, \epsilon_{1U}]$, $\epsilon_{0N}\epsilon_{1N} \in [\epsilon_{0L}\epsilon_{1L}, \epsilon_{0U}\epsilon_{1U}]$, $\epsilon_{0N}^2 \in [\epsilon_{0L}^2, \epsilon_{0U}^2]$, $\epsilon_{1N}^2 \in [\epsilon_{1L}^2, \epsilon_{1U}^2]$.

$$C_{xN} = \frac{S_{xN}}{\bar{X}_N},\ C_{yN} = \frac{S_{yN}}{\bar{Y}_N},\ C_{yxN} = \rho_{yxN} C_{yN} C_{xN},$$

where, $\theta_N = \frac{1}{n_N} - \frac{1}{N_N}$; $\theta_N \in [\theta_L, \theta_U]$, $n_N \in [n_L, n_U]$.

$$\rho_{yxN} = \frac{S_{yxN}}{S_{yN} S_{xN}},\ \rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}],$$

$$S_{yxN} \in [S_{yxL}, S_{yxU}].$$

Based on the above notations and terminologies, the general classes of neutrosophic estimators are formulated and discussed in the following sections.

2. GENERAL CLASS OF NEUTROSOPHIC ESTIMATORS

Building upon the formulation introduced by Srivastava (1971), we extend the classical framework to the neutrosophic setting and define the following general class of ratio-type estimators:

$$\bar{y}_{classN} = \bar{y}_N g(u_N) \tag{2.1}$$

where $g(u_N)$, for $u_N = \frac{\bar{x}_N}{\bar{X}_N}$, represents a parametric function satisfying the regularity conditions given below:

i) $g(1) = 1$ (2.2)

ii) The existence of the first and second partial derivatives of $g(u_N)$ with respect to u_N is assumed.

The values of γ_N and δ_N are taken as known constants, where:

$$\gamma_N = \left.\frac{\partial g(u_N)}{\partial u_N}\right|_{u_N=1} \text{ and } \delta_N = \left.\frac{\partial^2 g(u_N)}{\partial u_N^2}\right|_{u_N=1} \tag{2.3}$$

Applying the Taylor series expansion to the considered function yields the following approximate result:

$$\bar{y}_{classN} = \bar{y}_N \left[g(1) + (u_N - 1) \frac{\partial g(u_N)}{\partial u_N}\bigg|_{u_N=1} + \frac{(u_N-1)^2}{2} \frac{\partial^2 g(u_N)}{\partial u_N^2}\bigg|_{u_N=1} + \ldots \right] \quad (2.4)$$

On using the results from equation (2.3) and equation (2.4), we have:

$$\bar{y}_{classN} = \bar{y}_N \left[1 + (u_N - 1)\gamma_N + \frac{(u_N-1)^2}{2}\delta_N + \ldots \right] \quad (2.5)$$

$$\bar{y}_{classN} = \bar{Y}_N (1 + \epsilon_{0N}) \left[1 + \epsilon_{1N}\gamma_N + \frac{\epsilon_{1N}^2}{2}\delta_{iN} + \ldots \right] \quad (2.6)$$

By expanding equation (2.6) and retaining terms up to the $O_p\left(n^{-1}\right)$, we get:

$$\bar{y}_{classN} = \bar{Y}_N \left[1 + \epsilon_{0N} + \epsilon_{1N}\gamma_N + \frac{\epsilon_{1N}^2}{2}\delta_N + \epsilon_{0N}\epsilon_{1N}\gamma_N \ldots \right] \quad (2.7)$$

The MSE associated with the estimator class $\bar{y}_{classN}$ is given by:

$$MSE\left(\bar{y}_{classN}\right) = E\left[\bar{y}_{classN} - \bar{Y}_N\right]^2 \quad (2.8)$$

$$MSE\left(\bar{y}_{classN}\right) \cong E\left[\bar{Y}_N \left\{ 1 + \epsilon_{0N} + \epsilon_{1N}\gamma_N + \frac{\epsilon_{1N}^2}{2}\delta_N + \epsilon_{0N}\epsilon_{1N}\gamma_N + \ldots \right\} - \bar{Y}_N \right]^2 \quad (2.9)$$

$$MSE\left(\bar{y}_{classN}\right) \cong \bar{Y}_N^2 E\left(\epsilon_{0N} + \epsilon_{1N}\gamma_N\right)^2 \quad (2.10)$$

$$MSE\left(\bar{y}_{classN}\right) \cong \bar{Y}_N^2 E\left[\epsilon_{0N}^2 + \epsilon_{1N}^2\gamma_N^2 + 2\epsilon_{0N}\epsilon_{1N}\gamma_N\right] \quad (2.11)$$

$$MSE\left(\bar{y}_{classN}\right) \cong \bar{Y}_N^2 \theta_N \left[C_{yN}^2 + C_{xN}^2\gamma_N^2 + 2\gamma_N\rho_{xyN}C_{yN}C_{xN}\right] \quad (2.12)$$

$$MSE\left(\bar{y}_{classN}\right) \cong \theta_N \left[S_{yN}^2 + S_{xN}^2\gamma_N^2 + 2\gamma_N S_{xyN}\right] \quad (2.13)$$

By taking the derivative of equation (2.13) with respect to γ_N and setting the derivative to zero, we get:

$$\gamma_{optN} = -\frac{S_{xyN}}{S_{xN}^2}. \quad (2.14)$$

By replacing γ_N in equation (2.14) with its optimum value, we obtain the minimum MSE:

$$MSE\left(\bar{y}_{classN}\right)_{min} = \theta_N S_{yN}^2 \left(1 - \rho_{xyN}^2\right). \quad (2.15)$$

2.1 Neutrosophic Ratio Estimator

Following the classical ratio estimator proposed by Cochran (1940), Tahir *et al.* (2021) extended the formulation to the neutrosophic framework and proposed the following ratio estimator of the finite population mean:

$$\bar{y}_{RN} = \bar{y}_N \left(\frac{\bar{X}_N}{\bar{x}_N} \right) \quad (2.1.1)$$

We have

$g(u_N) = u_N^{-1}$ with

$\gamma_N = -1$ and $\delta_N = 2$.

2.2 Neutrosophic Product Estimator

Following the classical product estimator proposed by Murthy (1964), Vishwakarma and Singh (2022) extended the formulation to the neutrosophic framework and proposed the following neutrosophic product estimator:

$$\bar{y}_{PN} = \bar{y}_N \left(\frac{\bar{x}_N}{\bar{X}_N} \right) \quad (2.2.1)$$

We have

$g(u_N) = u_N$ with

$\gamma_N = 1$ and $\delta_N = 0$.

2.3 Sisodia and Dwivedi (1981) Type Neutrosophic Estimator

Inspired by the modified ratio estimation approach of Sisodia and Dwivedi (1981), Tahir *et al.* (2021) introduced a neutrosophic ratio-based estimator, given by:

$$\bar{y}_{SDN} = \bar{y}_N \left(\frac{\bar{X}_N + C_{xN}}{\bar{x}_N + C_{xN}} \right) \quad (2.3.1)$$

In the estimator in equation (2.3.1), we have

$$g(u_N)=\left(1+\frac{C_{xN}}{\overline{X}_N}\right)\left(u_N+\frac{C_{xN}}{\overline{X}_N}\right)^{-1} \text{ with}$$

$$\gamma_N=-\left(1+\frac{C_{xN}}{\overline{X}_N}\right)^{-1};\ \delta_N=2\left(1+\frac{C_{xN}}{\overline{X}_N}\right)^{-2}.$$

It should be noted that, with this particular choice of γ_N and δ_N, the resulting estimator class does not attain minimum variance. Consequently, the estimator proposed by Sisodia and Dwivedi (1981) can't outperform the neutrosophic linear regression estimator.

2.4 Upadhyaya and Singh (1999) Type Neutrosophic Estimator

Based on the formulation of Upadhyaya and Singh (1999), Tahir *et al.* (2021) proposed the following neutrosophic ratio-type estimator incorporating the CV and kurtosis:

$$\overline{y}_{USN}=\overline{y}_N\left(\frac{\overline{X}_N\beta_{2(x)N}+C_{xN}}{\overline{x}_N\beta_{2(x)N}+C_{xN}}\right) \tag{2.4.1}$$

For the estimator given in equation (2.4.1), we have:

$$g(u_N)=\frac{\overline{X}_N\beta_{2(x)N}+C_{xN}}{\overline{X}_N\beta_{2(x)N}u_N+C_{xN}} \text{ with}$$

$$\gamma_N=-\frac{\overline{X}_N\beta_{2(x)N}}{\overline{X}_N\beta_{2(x)N}+C_{xN}};\ \delta_N=\frac{2\left(\overline{X}_N\beta_{2(x)N}\right)^2}{\left(\overline{X}_N\beta_{2(x)N}+C_{xN}\right)^2}.$$

The chosen values of γ_N and δ_N do not yield the minimum MSE within this general class of estimators.

2.5 Neutrosophic Exponential Estimators

Following Bahl and Tuteja (1991), Tahir *et al.* (2021) extended the exponential ratio estimator to the neutrosophic framework, while Yadav and Prasad (2024) extended the corresponding exponential product estimator as follows:

$$\overline{y}_{BT_{1N}}=\overline{y}_N exp\left(\frac{\overline{X}_N-\overline{x}_N}{\overline{X}_N+\overline{x}_N}\right) \tag{2.5.1}$$

In the formulation specified by equation (2.5.1), we obtain the following values:

$$g(u_N)=exp\left(\frac{1-u_N}{1+u_N}\right) \text{ with}$$

$$\gamma_N=-\frac{1}{2};\ \delta_N=\frac{3}{4}.$$

The formulation of Bahl and Tuteja (1991) leads to the following neutrosophic product-type estimator:

$$\overline{y}_{BT_{2N}}=\overline{y}_N exp\left(\frac{\overline{x}_N-\overline{X}_N}{\overline{X}_N+\overline{x}_N}\right) \tag{2.5.2}$$

In the formulation specified by equation (2.5.2), we obtain the following values:

$$g(u_N)=exp\left(\frac{u_N-1}{1+u_N}\right) \text{ with}$$

$$\gamma_N=\frac{1}{2};\ \delta_N=-\frac{1}{4}.$$

It may be noted that the prescribed values of γ_N and δ_N do not yield the minimum attainable MSE for the general estimator class.

2.6 Neutrosophic generalized exponential-type estimator

Building on the generalized exponential-type estimator of Khan *et al.* (2014), Tahir *et al.* (2021) proposed its neutrosophic extension aimed at estimating the population mean as follows:

$$\overline{y}_{EN}=\overline{y}_N exp\left[\alpha_N\left(\frac{\overline{X}_N^{\frac{1}{h}}-\overline{x}_N^{\frac{1}{h}}}{\overline{X}_N^{\frac{1}{h}}+(a_N-1)\overline{x}_N^{\frac{1}{h}}}\right)\right] \tag{2.6.1}$$

where α_N $(-\infty<\alpha_N<\infty)$ is a known real constant, while $a_N(a_N\neq 0)$ is an unknown constant to be determined such that $\overline{y}_{EN}$ attains optimality and its MSE is minimized.

$$g(u_N)=\overline{y}_N exp\left[\alpha_N\left(\frac{1-u_N^{\frac{1}{h}}}{1+(a-1)u_N^{\frac{1}{h}}}\right)\right] \text{ with}$$

$$\gamma_N=-\frac{\alpha_N}{ha_N};\ \delta_N=\frac{\alpha_N}{h^2\alpha^2}\left[a_N(1+h)-2+\alpha_N\right].$$

When a_N takes its optimal value, the first-order approximation to the minimum MSE of $\overline{y}_{EN}$ can be expressed as:

$$MSE\left(\overline{y}_{EN}\right)_{Min} = \theta_N S_{yN}^2\left(1-\rho_{xyN}^2\right).$$

The resulting MSE has the same value as that of the linear regression estimator.

2.7 Subramani and Kumarapandiyan (2012) Type Neutrosophic Estimator

Following Subramani and Kumarapandiyan (2012), Yadav and Smarandache (2023) proposed the following neutrosophic ratio estimator incorporating the median and CV of the auxiliary variable:

$$\overline{y}_{SKN} = \overline{y}_N\left(\frac{\overline{X}_N C_{xN} + M_{d_N}}{\overline{x}_N C_{xN} + M_{d_N}}\right) \tag{2.7.1}$$

In the formulation specified by equation (2.7.1), we obtain the following values:

$$g\left(u_N\right) = \frac{\overline{X}_N C_{xN} + M_{d_N}}{\overline{X}_N C_{xN} u_N + M_{d_N}};\ \gamma_N = -\frac{\overline{X}_N C_{xN}}{\overline{X}_N C_{xN} + M_{d_N}};$$

$$\delta_N = \frac{2\left(\overline{X}_N C_{xN}\right)^2}{\left(\overline{X}_N C_{xN} + M_{d_N}\right)^2}.$$

It may be observed that the specified choice of γ_N and δ_N does not result in the minimum MSE for the general class of estimators.

2.8 Kadilar (2016) Type Neutrosophic Estimator

Adapting the formulation of Kadilar (2016) to the neutrosophic paradigm, Yadav and Prasad (2024) proposed the following modified exponential-type estimator:

$$\overline{y}_{KN} = \overline{y}_N\left(\frac{\overline{x}_N}{\overline{X}_N}\right)^{\alpha_N} exp\left(\frac{\overline{X}_N - \overline{x}_N}{\overline{X}_N + \overline{x}_N}\right) \tag{2.8.1}$$

where α_N is selected to attain the minimum mean squared error of the estimator $\overline{y}_{KN}$.

In the formulation given by equation (2.8.1), the relevant parameters take the following values:

$$g\left(u_N\right) = \left(u_N\right)^{\alpha_N} exp\left(\frac{1-u_N}{1+u_N}\right) \text{ with}$$

$$\gamma_N = \alpha_N - \frac{1}{2};\ \delta_N = \alpha_N^2 - 2\alpha_N + \frac{3}{4}.$$

When a_N takes its optimal value, the first-order approximation to the minimum MSE of $\overline{y}_{KN}$ can be expressed as:

$$MSE\left(\overline{y}_{KN}\right)_{Min} = \theta_N S_{yN}^2\left(1-\rho_{xyN}^2\right).$$

The resulting MSE is equal to the linear regression estimator.

2.9 Singh *et al.* (2025) Estimator

Singh and Espejo (2003) proposed a combined ratio-product estimation framework, which was subsequently formulated in the neutrosophic setting by Singh *et al.* (2025) as the following estimator:

$$\overline{y}_{B1N} = \overline{y}_N\left[k_{1N}\left(\frac{\overline{X}_N}{\overline{x}_N}\right) + \left(1-k_{1N}\right)\left(\frac{\overline{x}_N}{\overline{X}_N}\right)\right] \tag{2.9.1}$$

where the value of k_{1N} is determined by minimizing the MSE of the estimator.

For the estimator specified in equation (2.9.1), we get

$$g\left(u_N\right) = \frac{k_{1N}}{u_N} + \left(1-k_{1N}\right)u_N \text{ with}$$

$$\gamma_N = 1-2k_{1N}\ ;\ \delta_N = 2k_{1N}.$$

When k_{1N} is set to its optimum, the estimator attains a first-order approximate minimum MSE identical to the linear regression estimator.

Singh *et al.* (2008) suggested a combined ratio-product exponential estimation framework, which was subsequently formulated in the neutrosophic setting by B. Singh *et al.* (2025) as the following estimator:

$$\overline{y}_{B2N} = \overline{y}_N\left[k_{2N} exp\left(\frac{\overline{X}_N - \overline{x}_N}{\overline{X}_N + \overline{x}_N}\right) + \left(1-k_{2N}\right) exp\left(\frac{\overline{x}_N - \overline{X}_N}{\overline{X}_N + \overline{x}_N}\right)\right] \tag{2.9.2}$$

where the value of k_{2N} is determined by minimizing the MSE of the estimator.

For the estimator specified in equation (2.9.2), we have

$$g(u_N) = k_{2N} exp\left(\frac{1-u_N}{1+u_N}\right) + (1-k_{2N}) exp\left(\frac{u_N-1}{1+u_N}\right)$$

with

$$\gamma_N = \frac{1}{2} - k_{2N};\ \delta_N = k_{2N} - \frac{1}{4}.$$

When k_{2N} is set to its optimum, the estimator attains a first-order approximate minimum MSE identical to the linear regression estimator.

3. A BROADER CLASS OF NEUTROSOPHIC ESTIMATORS

Following the formulation of Singh and Upadhyaya (1986), a broader class of neutrosophic estimators for $\overline{Y}_N$ is given by:

$$\overline{y}_{BclassN} = \overline{y}_N h(\overline{y}_N, u_N) \tag{3.1}$$

where $h(\overline{y}_N, u_N)$ denotes a parametric function satisfying the regularity conditions given below:

i) $h(\overline{Y}_N, 1) = \psi_{1N}\overline{Y}_N$, where ψ_{1N} represents a fixed constant.

ii) Since h has first and second-order partial derivatives, its approximation is written as:

$$h(\overline{y}_N, u_N) = h\left[\overline{Y}_N + (\overline{y}_N - \overline{Y}_N), 1 + (u_N - 1)\right] \tag{3.2}$$

$$= \left[h(\overline{Y}_N, 1) + (\overline{y}_N - \overline{Y}_N)\frac{\partial h}{\partial \overline{y}_N}\bigg|_{\overline{y}_N = \overline{Y}_N} + (u_N - 1)\frac{\partial h}{\partial u_N}\bigg|_{u_N = 1} + \right.$$

$$\frac{(\overline{y}_N - \overline{Y}_N)^2}{2}\frac{\partial^2 h}{\partial \overline{y}_N^2}\bigg|_{\overline{y}_N = \overline{Y}_N} + \frac{(u_N - 1)^2}{2}\frac{\partial^2 h}{\partial u_N^2}\bigg|_{u_N = 1} +$$

$$\left. (\overline{y}_N - \overline{Y}_N)(u_N - 1)\frac{\partial^2 h}{\partial u_N \partial \overline{y}_N}\bigg|_{u_N = 1,\ \overline{y}_N = \overline{Y}_N} + \ldots \right] \tag{3.3}$$

$$= \left[h(\overline{Y}_N, 1) + (\overline{y}_N - \overline{Y}_N)\psi_{1N} + (u_N - 1)\psi_{2N} + \right.$$

$$\frac{(\overline{y}_N - \overline{Y}_N)^2}{2}\frac{\partial^2 h}{\partial \overline{y}_N^2}\bigg|_{\overline{y}_N = \overline{Y}_N} + \frac{(u_N - 1)^2}{2}\frac{\partial^2 h}{\partial u_N^2}\bigg|_{u_N = 1} +$$

$$\left. (\overline{y}_N - \overline{Y}_N)(u_N - 1)\frac{\partial^2 h}{\partial u_N \partial \overline{y}_N}\bigg|_{u_N = 1,\ \overline{y}_N = \overline{Y}_N} + \ldots \right] \tag{3.4}$$

$$= \psi_{1N}\overline{y}_N + (u_N - 1)\psi_{2N} + O_p\left(n^{-1}\right) \tag{3.5}$$

The optimum values of ψ_{1N} and ψ_{2N} are obtained by minimizing:

$$MSE(\overline{y}_{BclassN}) = E\left[\psi_{1N}\overline{y}_N + (u_N - 1)\psi_{2N} - \overline{Y}_N\right]^2 \tag{3.6}$$

The optimal values of ψ_{1N} and ψ_{2N} are obtained as follows:

$$\psi_{1N} = \frac{1}{1 + \theta_N C_{yN}^2\left(1 - \rho_{xyN}^2\right)} \tag{3.7}$$

$$\psi_{2N} = -\frac{\overline{Y}_N \rho_{xyN} C_{yN}}{C_{xN}\left\{1 + \theta_N C_{yN}^2\left(1 - \rho_{xyN}^2\right)\right\}} \tag{3.8}$$

The broader class of estimators attains its minimum MSE, given by:

$$MSE(\overline{y}_{BclassN})_{Min} = \frac{\theta_N \overline{Y}_N^2 C_{yN}^2\left(1 - \rho_{xyN}^2\right)}{1 + \theta_N C_{yN}^2\left(1 - \rho_{xyN}^2\right)} \tag{3.9}$$

The result in equation (3.9) demonstrates that the proposed broader class of estimators, formulated as a neutrosophic extension of the estimator introduced by Singh and Upadhyaya (1986), provides greater efficiency than the conventional linear regression estimator. Furthermore, the neutrosophic estimator motivated by Srivastava (1980) can be obtained as a special case of the proposed neutrosophic extension of the Singh and Upadhyaya (1986) class by setting $\psi_{1N} = 1$.

We now present several estimators obtained as specific forms within the proposed broader class of neutrosophic estimators.

3.1 Neutrosophic Regression Estimator

Following Hansen *et al.* (1953), Singh *et al.* (2023) proposed the following neutrosophic regression estimator:

$$\overline{y}_{lrN} = \overline{y}_N + \hat{\beta}_N\left(\overline{X}_N - \overline{x}_N\right) \tag{3.1.1}$$

where $\hat{\beta}_N = \frac{s_{xyN}}{s_{xN}^2}$ indicates the neutrosophic sample-based regression coefficient.

Whenever a non-zero correlation exists between the study variable and the auxiliary variable, the regression estimator offers higher efficiency than $\bar{y}_N$.

The linear regression estimator attains its minimum variance, given by:

$$V\left(\bar{y}_{lrN}\right)_{Min} = \theta_N S_{yN}^2\left(1-\rho_{xyN}^2\right). \tag{3.1.2}$$

3.2 Gupta and Shabbir (2008) Type Neutrosophic Estimators

Inspired by Gupta and Shabbir (2008), We propose the neutrosophic estimator class given below:

$$t_{GS_N} = \left[K_{1N}\bar{y}_N + K_{2N}\left(\bar{X}_N - \bar{x}_N\right)\right]\left(\frac{\alpha_N \bar{X}_N + \beta_N}{\alpha_N \bar{x}_N + \beta_N}\right) \tag{3.2.1}$$

Here, K_{1N}, K_{2N}, α_N, and β_N are constants, and the minimum MSE of t_{GS_N} is equal to that of the broader estimator class.

Equation (3.2.1) can be represented in the following form:

$$t_{GS_N} = h\left(\bar{y}_N, u_N\right)$$
$$= \left[K_{1N}\bar{y}_N + K_{2N}\bar{X}_N\left(1-u_N\right)\right]\left(\frac{\alpha_N \bar{X}_N + \beta_N}{\alpha_N \bar{X}_N u_N + \beta_N}\right) \tag{3.2.2}$$

Where $h\left(\bar{Y}_N, 1\right) = K_{1N}\bar{Y}_N$

$$\psi_{1N} = \left.\frac{\partial h\left(\bar{y}_N, u_N\right)}{\partial \bar{y}_N}\right|_{\left(\bar{Y}_N, 1\right)} = 1 \tag{3.2.3}$$

$$\psi_{2N} = \left.\frac{\partial h\left(\bar{y}_N, u_N\right)}{\partial u_N}\right|_{\left(\bar{Y}_N, 1\right)} = -K_{2N}\bar{X}_N - \frac{\alpha_N \bar{X}_N \bar{Y}_N}{\alpha_N \bar{X}_N + \beta_N} \tag{3.2.4}$$

$$MSE\left(t_{GS_N}\right)_{Min} = \frac{\theta_N \bar{Y}_N^2 C_{yN}^2\left(1-\rho_{xyN}^2\right)}{1+\theta_N C_{yN}^2\left(1-\rho_{xyN}^2\right)} \tag{3.2.5}$$

The estimator $\left(t_{GS_N}\right)$ attains the same minimum MSE as the broader class of estimators.

3.3 Singh and Kumari (2024) Estimator

Inspired by Mishra *et al.* (2017), Singh and Kumari (2024) developed a neutrosophic estimator, given by:

$$t_{SK_{1N}} = \bar{y}_N\left(p_{1N}+1\right) + p_{2N} log\left(\frac{\bar{x}_N}{\bar{X}_N}\right) \tag{3.3.1}$$

where p_{1N} and p_{2N} represent constants determined by minimizing the MSE of the estimators.

The estimator given in equation (3.3.1) can be expressed as:

$$t_{SK_{1N}} = h\left(\bar{y}_N, u_N\right) = \bar{y}_N\left(p_{1N}+1\right) + p_{2N} log\left(u_N\right) \tag{3.3.2}$$

Here, $h\left(\bar{Y}_N, 1\right) = \bar{Y}_N\left(p_{1N}+1\right)$

$$\psi_{1N} = \left.\frac{\partial h\left(\bar{y}_N, u_N\right)}{\partial \bar{y}_N}\right|_{\left(\bar{Y}_N, 1\right)} = 1 + p_{1N} \tag{3.3.3}$$

$$\psi_{2N} = \left.\frac{\partial h\left(\bar{y}_N, u_N\right)}{\partial u_N}\right|_{\left(\bar{Y}_N, 1\right)} = p_{2N} \tag{3.3.4}$$

Inspired by Mishra *et al.* (2017), Singh and Kumari (2024) introduced a neutrosophic exponential-cum-logarithmic type estimator, given by:

$$t_{SK_{2N}} = p_{3N}\bar{y}_N + p_{4N} exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right)\left[1 + log\left(\frac{\bar{x}_N}{\bar{X}_N}\right)\right] \tag{3.3.5}$$

where p_{3N} and p_{4N} represent constants determined by minimizing the MSE of the estimators ($t_{SK_{2N}}$).

The estimator given in equation (3.3.5) can be expressed as:

$$t_{SK_{2N}} = h\left(\bar{y}_N, u_N\right)$$
$$= p_{3N}\bar{y}_N + p_{4N} exp\left(\frac{u_N - 1}{u_N + 1}\right)\left[1 + log\left(u_N\right)\right] \tag{3.3.6}$$

Where $h\left(\bar{Y}_N, 1\right) = p_{3N}\bar{Y}_N + p_{4N}$

$$\psi_{1N} = \left.\frac{\partial h\left(\bar{y}_N, u_N\right)}{\partial \bar{y}_N}\right|_{\left(\bar{Y}_N, 1\right)} = p_{3N} \tag{3.3.7}$$

$$\psi_{2N} = \left.\frac{\partial h\left(\bar{y}_N, u_N\right)}{\partial u_N}\right|_{\left(\bar{Y}_N, 1\right)} = \frac{3}{2}\left(1-p_{3N}\right)\bar{Y}_N \tag{3.3.8}$$

When the weights satisfy the constraint $(p_{3N} + p_{4N} = 1)$, the estimator $t_{SK_{2N}}$ attains the same minimum MSE as the linear regression estimator.

3.4 Subramani and Kumarapandiyan (2012) Type Neutrosophic Estimator

We have adapted the estimator introduced by Subramani and Kumarapandiyan (2012) to the neutrosophic setting, which is given by:

$$t_{SK3N} = \frac{\bar{y}_N + \beta\left(\bar{X}_N - \bar{x}_N\right)}{\bar{x}_N + M_{dN}}\left(\bar{X}_N + M_{dN}\right) \tag{3.4.1}$$

The estimator specified by equation (3.4.1) can be rewritten as:

$$t_{SK3N} = h\left(\bar{y}_N, u_N\right) = \frac{\bar{y}_N + \beta\bar{X}_N\left(1-u_N\right)}{\bar{x}_N + M_{dN}}\left(\bar{X}_N + M_{dN}\right) \tag{3.4.2}$$

Where $h\left(\bar{Y}_N, 1\right) = \bar{Y}_N$

$$\psi_{1N} = \left.\frac{\partial h\left(\bar{y}_N, u_N\right)}{\partial \bar{y}_N}\right|_{\left(\bar{Y}_N, 1\right)} = 1 \tag{3.4.3}$$

$$\psi_{2N} = \left.\frac{\partial h\left(\bar{y}_N, u_N\right)}{\partial u_N}\right|_{\left(\bar{Y}_N, 1\right)} = -\beta_N\bar{X}_N - \frac{\bar{Y}_N\bar{X}_N}{\bar{X}_N + M_{dN}} \tag{3.4.4}$$

It follows that the estimator defined in equation (3.4.1) constitutes a particular form of both the Srivastava (1980) and Singh and Upadhyaya (1986) classes of neutrosophic estimators.

3.5 Singh *et al.* (2025) Estimator

Following Grover and Kaur (2014), Singh *et al.* (2025) developed a neutrosophic difference-cum-exponential estimator for indeterminate conditions, expressed as:

$$t_{S1_N} = \left[K_{1N}\bar{y}_N + K_{2N}\left(\bar{X}_N - \bar{x}_N\right)\right]\exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right) \tag{3.5.1}$$

In this formulation, K_{1N} and K_{2N} are two real constants to be estimated.

The estimator (3.5.1) can be expressed as:

$$t_{S1_N} = h\left(\bar{y}_N, u_N\right) = \left[K_{1N}\bar{y}_N + K_{2N}\bar{X}_N\left(1-u_N\right)\right]\exp\left(\frac{1-u_N}{1+u_N}\right) \tag{3.5.2}$$

Where $h\left(\bar{Y}_N, 1\right) = K_{1N}\bar{Y}_N$

$$\psi_{1N} = \left.\frac{\partial h\left(\bar{y}_N, u_N\right)}{\partial \bar{y}_N}\right|_{\left(\bar{Y}_N, 1\right)} = 1 \tag{3.5.3}$$

$$\psi_{2N} = \left.\frac{\partial h\left(\bar{y}_N, u_N\right)}{\partial u_N}\right|_{\left(\bar{Y}_N, 1\right)} = -K_{2N}\bar{X}_N - \frac{1}{2}\bar{Y}_N. \tag{3.5.4}$$

The minimum MSE achieved by the estimator t_{S1_N} is equal to that obtained for the broader class of estimators.

3.6 Singh and Tiwari (2025) Estimator

The logarithmic-type neutrosophic estimator proposed by Singh and Tiwari (2025), which incorporates the median of the auxiliary variable, is given by:

$$t_{ST_N} = \bar{y}_N + W_N log\left(\frac{\bar{X}_N + M_{dN}}{\bar{x}_N + M_{dN}}\right) \tag{3.6.1}$$

The value of W_{1N} is chosen so as to minimize the MSE.

$$t_{ST_N} = h\left(\bar{y}_N, u_N\right) = \bar{y}_N + W_N log\left(\frac{1 + \frac{M_{dN}}{\bar{X}_N}}{u_N + \frac{M_{dN}}{\bar{X}_N}}\right) \tag{3.6.2}$$

Where $h\left(\bar{Y}_N, 1\right) = \bar{Y}_N$

$$\psi_{1N} = \left.\frac{\partial h\left(\bar{y}_N, u_N\right)}{\partial \bar{y}_N}\right|_{\left(\bar{Y}_N, 1\right)} = 1 \tag{3.6.3}$$

$$\psi_{2N} = \left.\frac{\partial h\left(\bar{y}_N, u_N\right)}{\partial u_N}\right|_{\left(\bar{Y}_N, 1\right)} = \frac{-K_{1N}\bar{X}_N}{\bar{X}_N + M_{dN}}. \tag{3.6.4}$$

For optimum value of W_N, the estimator attains a first-order approximate minimum MSE identical to the linear regression estimator.

4. A UNIFIED CLASS OF ESTIMATORS

Following Christiansen *et al.* (2026), we consider a wider class of estimators for the population mean $\overline{Y}_N$ defined as:

$$t^{*}_{Wclass} = \left(w_{1N}\overline{y}_N + w_{2N}\right)H\left(u_N\right) \tag{4.1}$$

with $u_N = \dfrac{\overline{x}_N}{\overline{X}_N}$.

$H\left(u_N\right)$ is any continuous and differentiable function of u_N.

This class is quite flexible and includes several well-known estimators as special cases. It is worth noting that for certain choices of $H\left(u_N\right)$, the estimator may suffer from computational discontinuity at $u_N = 1$ (i.e., when $\overline{x}_N = \overline{X}_N$), requiring a patch to make it operational. Below, we discuss some members of this class.

4.1 Shakoor *et al.* (2026) Type Neutrosophic Estimator

Following Shakoor *et al.* (2026), we have suggested the following log-ratio type neutrosophic estimators belonging to the above class. The first estimator is defined as:

$$t_{S2N} = \left(w_{1N}\overline{y}_N + w_{2N}\right)\left(\frac{\overline{X}_N}{\overline{x}_N - \overline{X}_N}\right)ln\left(\frac{\overline{x}_N}{\overline{X}_N}\right) \tag{4.1.1}$$

This corresponds to: $H\left(u_N\right) = \dfrac{\ln\left(u_N\right)}{u_N - 1}$, $u_N = \dfrac{\overline{x}_N}{\overline{X}_N}$.

However, the estimator has a computational discontinuity at $\overline{x}_N = \overline{X}_N$. To address this, a patched version is suggested:

$$t_{S2N} = \begin{bmatrix} \left(w_{1N}\overline{y}_N + w_{2N}\right)\left(\dfrac{\overline{X}_N}{\overline{x}_N - \overline{X}_N}\right)ln\left(\dfrac{\overline{x}_N}{\overline{X}_N}\right), & \overline{x}_N \neq \overline{X}_N \\ w_{1N}\overline{y}_N + w_{2N}, & \overline{x}_N = \overline{X}_N \end{bmatrix} \tag{4.1.2}$$

This estimator is a particular form of the wider class equation (4.1.1) and is biased unless $w_{1N} \to 1$ and $w_{2N} \to 0$.

The minimum MSE is given by:

$$MSE\left(t_{S2N}\right)_{min} = \overline{Y}_N\left[1 - \frac{A_1D_1^2 + B_1C_1^2 - 2C_1D_1E_1}{A1B1 - E_1^2}\right] \tag{4.1.3}$$

where, $A_1 = 1 + \theta_N\left\{C_{yN}^2 + \dfrac{11}{12}C_{xN}^2 - 2\rho_{yxN}C_{yN}C_{xN}\right\}$,

$B_1 = 1 + \dfrac{11}{12}\theta_N C_{xN}^2$, $C_1 = 1 + \theta_N\left\{\dfrac{1}{3}C_{xN}^2 - \dfrac{\rho_{yxN}C_{yN}C_{xN}}{2}\right\}$,

$D_1 = 1 + \dfrac{1}{3}\theta_N C_{xN}^2$, $E_1 = 1 + \theta_N\left\{\dfrac{11}{12}C_{xN}^2 - \rho_{yxN}C_{yN}C_{xN}\right\}$.

4.2 Shabbir *et al.* (2021) Type Neutrosophic Estimator

Motivated by Shabbir *et al.* (2021), we have suggested a simple class of exponential-ratio-type estimators employing conventional and nonconventional measures is presented below:

$$t_{S3N} = \left(K_{1N}\overline{y}_N + K_{2N}\right)\exp\left(\frac{a_N\left(\overline{X}_N - \overline{x}_N\right)}{a_N\left(\overline{X}_N + \overline{x}_N\right) + 2b_N}\right) \tag{4.2.1}$$

This corresponds to: $H\left(u_N\right) = \exp\left(\dfrac{a_N\overline{X}_N\left(1 - u_N\right)}{a_N\overline{X}_N\left(1 + u_N\right) + 2b_N}\right)$.

The quantities K_{1N} and K_{2N} are constants and a_N and b_N denote the known conventional and nonconventional measures associated with the auxiliary variable.

The minimum MSE is given by:

$$MSE\left(t_{S3N}\right)_{min} = \overline{Y}_N\left[1 - \frac{\begin{matrix} A_2D_2^2 + B_2C_2^2 - 2C_2D_2E_2 + \\ 2B_2C_2 + D_2^2 - 2D_2E_2 + B_2 \end{matrix}}{A_2B_2 - E_2^2 + B_2}\right] \tag{4.2.2}$$

where, $\psi_i = \dfrac{a_N\overline{X}_N}{a_N\overline{X}_N + b_N}$.

$$A_2 = \theta_N \left\{ C_{yN}^2 + \psi_i^2 C_{xN}^2 - 2\psi_i \rho_{yxN} C_{yN} C_{xN} \right\},$$

$$B_2 = 1 + \psi_i^2 \theta_N C_{xN}^2,$$

$$C_2 = \theta_N \left\{ \frac{3}{8} \psi_i^2 C_{xN}^2 - \frac{\psi_i \rho_{yxN} C_{yN} C_{xN}}{2} \right\},$$

$$D_2 = 1 + \frac{3}{8} \theta_N \psi_i^2 C_{xN}^2,$$

$$E_2 = 1 + \theta_N \left\{ \psi_i^2 C_{xN}^2 - \psi_i \rho_{yxN} C_{yN} C_{xN} \right\}.$$

5. THE PROPOSED NEUTROSOPHIC ALMOST UNBIASED CLASS OF ESTIMATORS

Let us consider a general class of estimators defined as:

$$t_{iN} = \overline{y}_N g_i\left(u_N\right),\ i = 0,1,2\,. \tag{5.1}$$

where $u_N = \dfrac{\overline{x}_N}{\overline{X}_N}$ and $g_i\left(u_N\right)$ are parametric functions subject to the conditions below:

$$g_i\left(1\right) = 1 \tag{5.2}$$

The function $g_i\left(u_N\right)$ has first and second-order partial derivatives with respect to u_N, and are continuous in the neighbourhood of $u_N = 1$. Let:

$$\gamma_{iN} = \left.\frac{\partial g_i\left(u_N\right)}{\partial u_N}\right|_{u_N=1},\ \delta_{iN} = \left.\frac{\partial^2 g_i\left(u_N\right)}{\partial u_N^2}\right|_{u_N=1} \tag{5.3}$$

are known constants. Following the filtration technique of Singh and Singh (1991, 1993a, 1993b), we define a linear variety of estimators as:

$$t_N^* = \sum_{i=0}^{2} \alpha_{iN} t_{iN} \in A_N \tag{5.4}$$

subject to the constraints:

$$\sum_{i=0}^{2} \alpha_{iN} = 1;\ \alpha_{iN} \in \mathbb{R} \tag{5.5}$$

and the bias filtration constraint:

$$\sum_{i=0}^{2} \alpha_{iN} B(t_{iN}) = 0 \tag{5.6}$$

where $B(t_{iN})$ *denotes the bias of the estimator* t_{iN} *up to the first order of approximation.*

To derive the MSE of the proposed class, we obtain the Taylor expansion of $g_i\left(u_N\right)$ at $u_N = 1$:

$$g_i\left(u_N\right) = g_i\left(1\right) + \left(u_N - 1\right) \left.\frac{\partial g_i\left(u_N\right)}{\partial u_N}\right|_{u_N=1} + \frac{\left(u_N - 1\right)^2}{2} \left.\frac{\partial^2 g_i\left(u_N\right)}{\partial u_N^2}\right|_{u_N=1} + O_p\left(n^{-1}\right) \tag{5.7}$$

Using equation (5.3) and equation (5.7), we get:

$$g_i\left(u_N\right) = 1 + \left(u_N - 1\right)\gamma_{iN} + \frac{\left(u_N - 1\right)^2}{2} \delta_{iN} + O_p\left(n^{-1}\right) \tag{5.8}$$

$$t_{iN} = \overline{Y}_N \left(1 + \epsilon_0\right) \left[1 + \epsilon_{1N}\gamma_{iN} + \frac{\epsilon_{1N}^2}{2} \delta_{iN} + O_p\left(n^{-1}\right) \right] \tag{5.9}$$

Expanding equation (5.9) and retaining terms up to order $O_p\left(n^{-1}\right)$, we get:

$$t_{iN} = \overline{Y}_N \left[1 + \epsilon_{0N} + \epsilon_{1N}\gamma_{iN} + \frac{\epsilon_{1N}^2}{2} \delta_{iN} + \epsilon_{0N}\epsilon_{1N}\gamma_{iN} + O_p\left(n^{-1}\right) \right] \tag{5.10}$$

Substituting equation (5.10) into equation (5.4), we have:

$$t_N^* = \overline{Y}_N \left[1 + \epsilon_{0N} + \epsilon_{1N} \sum_{i=0}^{2} \alpha_{iN}\gamma_{iN} + \frac{\epsilon_{1N}^2}{2} \sum_{i=0}^{2} \alpha_{iN}\delta_{iN} + \epsilon_{0N}\epsilon_{1N} \sum_{i=0}^{2} \alpha_{iN}\gamma_{iN} + O_p\left(n^{-1}\right) \right] \tag{5.11}$$

For notational convenience, let:

$$K_N = \sum_{i=0}^{2} \alpha_{iN}\gamma_{iN},\ L_N = \sum_{i=0}^{2} \alpha_{iN}\delta_{iN} \tag{5.12}$$

Then equation (4.11) can be written as:

$$t_N^* = \overline{Y}_N \left[1 + \epsilon_{0N} + \epsilon_{1N}K_N + \frac{\epsilon_{1N}^2}{2} L_N + \epsilon_{0N}\epsilon_{1N}K_N + O_p\left(n^{-1}\right) \right] \tag{5.13}$$

The bias of t_N^*, to the first order of approximation, is given by:

$$B\left(t_N^*\right)=\overline{Y}_N\left[\frac{E(\epsilon_{1N}^2)}{2}L_N+E\left(\epsilon_{0N}\epsilon_{1N}\right)K_N\right] \quad (5.14)$$

we get:

$$B\left(t_N^*\right)=\overline{Y}_N\theta_N\left[\frac{C_{xN}^2}{2}L_N+\rho_{xyN}C_{yN}C_{xN}K_N\right] \quad (5.15)$$

The bias filtration constraint given in equation (5.6) ensures that:

$$\frac{\theta_N C_{xN}^2}{2}L_N+\theta_N\rho_{xyN}C_{yN}C_{xN}K_N=0 \quad (5.16)$$

which implies that the bias of t_N^* *vanishes up to order* $O_p\left(n^{-1}\right)$.

Now using equation (5.13) and retaining terms up to order $O_p\left(n^{-1}\right)$, the mean squared error (MSE) of t_N^* is:

$$MSE\left(t_N^*\right)=\overline{Y}_N^2\,E\left(\epsilon_{0N}+\epsilon_{1N}K_N\right)^2 \quad (5.17)$$

$$MSE\left(t_N^*\right)=\overline{Y}_N^2E\left[\epsilon_{0N}^2+\epsilon_{1N}^2K_N^2+2\epsilon_{0N}\epsilon_{1N}K_N\right] \quad (5.18)$$

Expanding equation (5.18), we obtain:

$$MSE\left(t_N^*\right)=\overline{Y}_N^2\theta_N\left[C_{yN}^2+K_N^2C_{xN}^2+2K_N\rho_{xyN}C_{yN}C_{xN}\right] \quad (5.19)$$

To find the optimal value of K_N *that minimizes the MSE given in equation (5.19), we differentiate with respect to* K_N and equate to zero:

$$\frac{\partial MSE\left(t_N^*\right)}{\partial K_N}=\overline{Y}_N^2\theta_N\left[2K_NC_{xN}^2+2\rho_{xyN}C_{yN}C_{xN}\right]=0 \quad (5.20)$$

Solving equation (5.20), we get the optimal value of K_N *as:*

$$K_{N\,opt}=-\frac{\rho_{xyN}C_{yN}C_{xN}}{C_{xN}^2}=-\rho_{xyN}\frac{C_{yN}}{C_{xN}} \quad (5.21)$$

Substituting equation (5.21) into equation (5.19), we obtain the minimum MSE:

$$MSE\left(t_N^*\right)_{min}=\theta_N S_{yN}^2\left(1-\rho^2\right) \quad (5.22)$$

This MSE is identical to the MSE obtained from the linear regression estimator.

The optimal weights α_{i_N} *are obtained by solving the system of equations:*

$$\begin{cases}\sum_{i=0}^{2}\alpha_{iN}=1\\ \sum_{i=0}^{2}\alpha_{iN}\gamma_{iN}=K_{opt}\\ \sum_{i=0}^{2}\alpha_{iN}B\left(t_{iN}\right)=0\end{cases} \quad (5.23)$$

In matrix form, equation (5.23) can be written as:

$$\begin{bmatrix}1 & 1 & 1\\ \gamma_{0N} & \gamma_{1N} & \gamma_{2N}\\ B\left(t_{0N}\right) & B\left(t_{1N}\right) & B\left(t_{2N}\right)\end{bmatrix}\begin{bmatrix}\alpha_{0N}\\ \alpha_{1N}\\ \alpha_{2N}\end{bmatrix}=\begin{bmatrix}1\\ K_{opt}\\ 0\end{bmatrix} \quad (5.24)$$

Solving equation (5.24) gives the unique values of $\alpha_{0N},\alpha_{1N},\alpha_{2N}$ *provided that:*

$$\begin{bmatrix}1 & 1 & 1\\ \gamma_{0N} & \gamma_{1N} & \gamma_{2N}\\ B\left(t_{0N}\right) & B\left(t_{1N}\right) & B\left(t_{2N}\right)\end{bmatrix}\neq 0 \quad (5.25)$$

Now, we proceed to discuss a few estimators constructed under the neutrosophic almost unbiased estimation class.

5.1 Singh *et al.* (2024) Estimator

We first consider the almost unbiased estimator for the population mean using neutrosophic information, proposed by Singh *et al.* (2024), as given below.

$$t_{1N}^*=\alpha_{0N}\overline{y}_N+\alpha_{1N}\overline{y}_N\left(\frac{a_N\overline{X}_N+b_N}{a_N\overline{x}_N+b_N}\right)^{K_N}+\alpha_{2N}\overline{y}_N\exp\left[K_N\left(\frac{\left(a_N\overline{X}_N+b_N\right)-\left(a_N\overline{x}_N+b_N\right)}{\left(a_N\overline{X}_N+b_N\right)+\left(a_N\overline{x}_N+b_N\right)}\right)\right] \quad (5.1.1)$$

In this formulation, the parameter K_N assumes two distinct values: $K_N=1$ for ratio estimators and $K_N=-1$ for product estimators. Consider a_N and b_N are either known constants or neutrosophic functions associated with the supplementary variable X_N.

Considering the estimator in equation (4.1.1), $g_1(u_N)=\left(\frac{a_N\bar{X}_N+b_N}{a_N\bar{X}_Nu_N+b_N}\right)^{K_N}$, $g_1(1)=1$, with

$$\gamma_{1N}=-K_N\left(\frac{a_N\bar{X}_N}{a_N\bar{X}_N+b_N}\right) \text{ and}$$

$$\delta_{1N}=K_N(K_N+1)\left(\frac{a_N\bar{X}_N}{a_N\bar{X}_N+b_N}\right)^2.$$

we consider

$$g_2(u_N)=\exp\left[K_N\left(\frac{a_N\bar{X}_N(1-u_N)}{a_N\bar{X}_N(1+u_N)+2b_N}\right)\right], \text{ with}$$

$g_2(1)=1$, which gives the corresponding constants:

$$\gamma_{2N}=-K_N\left(\frac{a_N\bar{X}_N}{2\left(a_N\bar{X}_N+b_N\right)}\right) \text{ and}$$

$$\delta_{2N}=\frac{3K_N}{4}\left(\frac{a_N\bar{X}_N}{a_N\bar{X}_N+b_N}\right)^2.$$

The proposed framework yields the following particular forms:

1. For $g_0(u_N)=1$, we get the usual neutrosophic sample mean: $t_{0N}=\bar{y}_N$.
2. For $g_1(u_N)=\left(\frac{a_N\bar{X}_N+b_N}{a_N\bar{X}_Nu_N+b_N}\right)^{K_N}$, we get the generalized neutrosophic ratio estimator.
3. For
$$g_2(u_N)=\exp\left[K_N\left(\frac{a_N\bar{X}_N(1-u_N)}{a_N\bar{X}_N(1+u_N)+2b_N}\right)\right],$$
we get the generalized neutrosophic exponential estimator.

5.2 Singh *et al.* (2026) Estimator

Based on the formulation proposed by Singh *et al.* (2026),the neutrosophic almost unbiased exponential-cum-sine-type estimator is given by:

$$t_{2N}^{*}=\alpha_{0N}\bar{y}_N+\alpha_{1N}\bar{y}_N exp\left[sin\left(\frac{\bar{X}_N-\bar{x}_N}{\bar{X}_N+\bar{x}_N}\right)\right]+ \alpha_{2N}\bar{y}_N\exp\left[sin\left(\frac{\bar{x}_N-\bar{X}_N}{\bar{X}_N+\bar{x}_N}\right)\right] \quad (5.2.1)$$

In the estimator given in equation (4.2.1), $g_1(u_N)=exp\left[sin\left(\frac{1-u_N}{1+u_N}\right)\right]$, $g_1(1)=1$, with $\gamma_{1N}=-\frac{1}{2}$ and $\delta_{1N}=\frac{3}{4}$.

we consider $g_2(u_N)=\exp\left[sin\left(\frac{u_N-1}{1+u_N}\right)\right]$, with $g_2(1)=1$, which gives the corresponding constants $\gamma_{2N}=\frac{1}{2}$ and $\delta_{2N}=-\frac{1}{4}$.

5.3 Singh *et al.* (2025) Estimator

Following Singh *et al.* (2025), the neutrosophic almost unbiased exponential-cum-logarithmic-type estimator can be written in the following form:

$$t_{3N}^{*}=\alpha_{0N}\bar{y}_N+\alpha_{1N}\bar{y}_N\left(\frac{\bar{x}_N}{\bar{X}_N}\right)^{a_N}exp\left(\frac{b_N\left(\bar{x}_N-\bar{X}_N\right)}{\left(\bar{X}_N+\bar{x}_N\right)}\right)+ \alpha_{2N}\bar{y}_N\left[1+log\left\{\frac{\bar{x}_N}{\bar{X}_N}\right\}\right] \quad (5.3.1)$$

In the estimator given in equation (4.3.1), $g_1(u_N)=u_N^{a_N}exp\left(\frac{b_N(u_N-1)}{1+u_N}\right)$, $g_1(1)=1$, with $\gamma_{1N}=a_N+\frac{b_N}{2}$ and $\delta_{1N}=a_N^2+a_Nb_N-a_N+\frac{b_N^2}{4}-\frac{b_N}{2}$.

we consider $g_2(u_N)=1+\log(u_N)$, with $g_2(1)=1$, which gives the corresponding constants $\gamma_{2N}=1$ and $\delta_{2N}=-1$.

6. EMPIRICAL STUDY

To evaluate the performance of the proposed estimators, an empirical study is conducted using two real-life datasets. Dataset 1 considers pre-pandemic networking percentage as the auxiliary variable and pre-pandemic sales percentage as the study variable. Dataset I was derived from data on medical product marketing and sales reported by Aleeswari *et al.* (2023). The networking and product sales information was gathered by medical professionals for two months before the lockdown (1 January-1 March 2021).

Dataset 2 consists of solar energy data containing indeterminate values, where temperature is treated as the neutrosophic auxiliary variable and next-day global horizontal irradiance as the neutrosophic study variable.

Dataset 1: In this study, the percentage of networking before the pandemic is considered as the auxiliary variable, whereas the percentage of sales before the pandemic is treated as the study variable, i.e.,

X_N: Pre-pandemic networking percentage Y_N: Pre-pandemic sales percentage

Dataset 2: The second dataset is based on real-life solar energy data from Aslam and Algarni (2020), containing indeterminate values. We consider next-day global horizontal irradiance as the neutrosophic variable of interest $Y_N \in [Y_L, Y_U]$ and temperature as the neutrosophic auxiliary variable $X_N \in [X_L, X_U]$.

X_N: Temperature Y_N: Next-day global horizontal irradiance

Table 1. Summary statistics of Datasets 1 and 2

Parameters	Dataset 1	Dataset 2
N_N	[30, 30]	[12, 12]
n_N	[9, 9]	[6, 6]
$\overline{Y}_N$	[37.0033, 37.3266]	[5209.667, 6312.417]
$\overline{X}_N$	[34.1600, 34.4500]	[27.200, 29.0083]
S_{yN}	[12.3153, 12.3233]	[1439.122, 1292.364]
S_{xN}	[15.0789, 15.0891]	[8.5090, 7.8829]
C_{yN}	[0.3328, 0.3301]	[0.2762, 0.2047]
C_{xN}	[0.4414, 0.4380]	[0.3128, 0.2717]
ρ_{yxN}	[0.8776, 0.8775]	[0.8215, 0.8441]

Percentage relative efficiency (PRE) is generally calculated in percentage terms by taking the ratio of the variance of the usual estimator to the MSE of the proposed or considered estimator. An estimator with a higher PRE is considered more efficient, as it provides estimates with greater precision and lower variability.

$$PRE = \frac{MSE(\overline{y}_N)}{MSE(estimator)} \times 100 .$$

Table 2. MSE values of the proposed and existing estimators for Datasets 1 and 2

Estimator	Dataset 1	Dataset 2
$\overline{y}_N$	[11.7951, 11.8082]	[172538.5, 139138]
$\overline{y}_{RN}$	[5.0857, 5.1003]	[72788.24, 72488.88]
$\overline{y}_{BT_{1N}}$	[3.2531, 3.2569]	[67339.55,44531.94]
$\overline{y}_{classN}$	[2.7110, 2.7166]	[56115.31, 40014.65]
$\overline{y}_{BclassN}$	[2.7054, 2.7105]	[55983.07, 39961.39]
t_{S2N}	[1.1798, 1.1832]	[17878.97, 17540.48]
t_{S3N}	[1.1092, 1.1130]	[16605.92, 16368.54]
t_N^*	[2.7110, 2.7166]	[56115.31,40014.65]

Table 3. PRE values of the proposed and existing estimators for Datasets 1 and 2

Estimator	PRE	PRE
$\overline{y}_N$	[100, 100]	[100, 100]
$\overline{y}_{RN}$	[231.9272, 231.5202]	[237.0417, 191.944]
$\overline{y}_{BT_{1N}}$	[362.5786, 362.5587]	[256.2217, 312.4454]
$\overline{y}_{classN}$	[435.0840, 434.6682]	[307.4714, 347.7177]
$\overline{y}_{BclassN}$	[435.9879, 435.6419]	[308.1977, 348.1811]
t_{S2N}	[999.7526, 998.0054]	[965.036, 793.2397]
t_{S3N}	[1063.3763, 1060.9497]	[1039.018, 850.0333]
t_N^*	[435.0840,434.6682]	[307.4714, 347.7177]

7. RESULTS AND DISCUSSION

The empirical findings in Tables 2 and 3 provide a clear comparison of neutrosophic estimators belonging to different classes, including the usual, ratio, exponential-ratio, regression, unified class of estimator, and almost-unbiased estimators. The usual neutrosophic estimator $(\overline{y}_N)$ has the largest MSE for both datasets and therefore serves as the least efficient benchmark. The ratio $(\overline{y}_{RN})$ and exponential-ratio estimators $\left(\overline{y}_{BT_{1N}}\right)$ substantially reduce the MSE, with the latter

showing better performance than the ratio estimator. The general regression $\left(\overline{y}_{classN}\right)$, broader $(\overline{y}_{BclassN})$ and almost-unbiased $\left(t_N^*\right)$ estimators provide further improvement, although $\overline{y}_{classN}$ and t_N^* yield identical results in the reported cases. Among these existing estimators, the estimator t_{S2N} and t_{S3N} estimators show particularly strong performance, with (t_{S3N}) achieving the lowest MSE among the existing competitors. The proposed estimators demonstrate competitive performance across both datasets, and the PRE values clearly indicate substantial efficiency gains over the usual estimator. Overall, the results show that the choice of estimator class has a notable effect on estimation efficiency, with the proposed class providing an effective alternative to the existing approaches.

8. CONCLUSION AND FUTURE RESEARCH DIRECTIONS

This review presents a unified perspective on several classes of estimators for estimating the finite population mean using auxiliary information under a neutrosophic framework. The main objective is to organize and describe different neutrosophic estimator structures proposed in the literature within suitable general classes. The review considers the general neutrosophic ratio-type class motivated by Srivastava (1971), the wider class associated with Singh and Upadhyaya (1986), the unified class of estimators introduced by Christiansen *et al.* (2026). In addition, an almost-unbiased class based on a linear combination of estimators has been formulated to provide a broader framework for representing and developing estimator families.

The unified representation demonstrates that several estimators proposed over the past decade can be viewed as particular members or special cases of the broader estimator classes considered in this paper. This organization establishes clear relationships among different estimator forms and provides a systematic framework for representing several ratio, product, exponential, logarithmic, and regression type neutrosophic estimators as particular forms of the broader classes suggested in this study. Overall, the unified framework provides a structured basis for studying auxiliary-information-based estimators under neutrosophic sampling. It may also help researchers identify relationships among apparently different estimators and formulate new estimator families without unnecessarily introducing completely independent forms.

Future research may focus on evaluating the proposed estimator classes through simulation and empirical studies under different population distributions, sample sizes, sampling fractions, auxiliary-variable correlations, and levels of neutrosophic uncertainty. The bias, mean squared error, consistency, asymptotic properties, and robustness of individual members may also be investigated. Further extensions can consider other sampling designs, such as stratified, systematic, cluster, and multistage sampling, as well as other finite-population parameters. In addition, attention may be given to computational stability and practical applicability when implementing these generalized estimator classes. Overall, such theoretical, simulation-based, and empirical investigations can help assess the behaviour and comparative performance of the individual estimators within the unified framework.

DECLARATIONS

- **Conflict of Interest**

  The author declares that he has no conflict of interest.

- **Data Availability**

  The data used in this study are secondary data obtained from publicly available sources of the Government of India. The data source and relevant details are provided in the manuscript.

- **Ethical Approval**

  Ethical approval was not required for this study because it did not involve human participants, animals, or collection of identifiable personal information.

- **Clinical Trial Number**

  Not Applicable. This study did not involve a clinical trial.

- **Consent To Publish**

  Not applicable. This study did not involve the recruitment or participation of human subjects.

- **Consent To Participate**

  This study did not involve the recruitment or participation of human subjects.

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